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Resolution and model building in the infinite-valued calculus of Łukasiewicz

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Abstract

We discuss resolution and its complexity in the infinite-valued sentential calculus of Łukasiewicz, with special emphasis on model building algorithms for satisfiable sets of clauses. We prove that resolution and model building are polynomially tractable in the fragments given by Horn clauses and by Krom clauses, i.e., clauses with at most two literals. Generalizing the pre-existing literature on resolution in infinite-valued logic, by a positive literal we mean a negationless formula in one variable, built only from the connectives $\oplus, \odot, \vee, \wedge$. We prove that the expressive power of our literals encompasses all monotone McNaughton functions of one variable.

Keywords: Resolution in infinite-valued logic; Łukasiewicz calculus; Polynomial complexity; Model building; Horn clauses, Krom clauses; Feasible automated deduction; Davis–Putnam procedure; McNaughton functions

1. Introduction

The infinite-valued sentential calculus of Łukasiewicz was introduced in [4] (see [11, Ch. IV]). Equivalence classes of n-variable propositions were characterized by McNaughton in [5] as piecewise linear continuous functions $f:[0,1]^n \to [0,1]$, each piece having integer coefficients.

As explained in [7], the classical two-valued calculus stands to the familiar game of Twenty Questions as the infinite-valued calculus stands to Ulam game of Twenty Questions with errors/lies. Owing to the importance of Ulam game for the logical treatment of partially erroneous information [9], where search strategies are regarded

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as model building procedures in the Łukasiewicz calculus, it becomes more and more important to develop efficient deductive algorithms for this calculus. This is our aim in this paper.

We shall discuss resolution, model building and their complexity. With respect to other approaches to resolution in infinite-valued logic [1, 2, 12] by a *positive literal* we mean a negationless formula in one variable built from the connectives $\oplus, \odot, \vee, \wedge$. Negative literals are then defined in the natural way, and so are clauses. We show that the expressive power of our literals encompasses all monotone McNaughton functions of one variable. Using our literals, the numerical (= nonlogical) part of the input can be restricted to specifying a single satisfiability threshold $\theta = c/d$, for suitable integers $1 \leqslant c \leqslant d$ in binary notation.

Thus, we say that a formula F is θ-satisfiable iff for some assignment (alias, model) α the truth-value of F is $\geqslant \theta$. We relate θ-satisfiability to the notion of satisfiability previously used by the first author in [6]. We prove the NP-completeness of the θ-satisfiability problem in the infinite-valued calculus – where the input is of the form (F, c, d), and the question is whether formula F is (c/d)-satisfiable. For the particular case when the input is a finite set S of clauses, we develop the appropriate notions of resolution and Davis–Putnam procedure (DPP), and prove that the latter is complete.

In the particular case when S is a finite set of Horn clauses, generalizing the classical results we provide decision and model building algorithms that are polynomial in $\log d$ and in the length of S.[2] We do the same in case S is a finite set of Krom clauses – i.e., clauses with at most two literals.

The only prerequisites to this paper are the rudiments of propositional logic, Turing complexity and topology. Unless otherwise specified, all Turing machines in this paper are *deterministic*.

2. Literals, clauses, satisfiability

Throughout this paper we shall make use of the alphabet Σ given by

$$\Sigma = \{X, |, \wedge, \vee, \odot, \oplus, \neg,), (\}. \tag{1}$$

Strings of the form $X|, X||, \ldots$ (in short, $X_1, X_2, \ldots$) are called *variables*. The set of *formulas* is defined by stipulating that each variable is a formula, and whenever two strings G and H are formulas then so are the strings $\neg G$, $(G \odot H)$, $(G \oplus H)$, $(G \wedge H)$, $(G \vee H)$. For the sake of readability we shall omit outer parentheses whenever possible.

By writing $F = F(X_1, \ldots, X_n)$ we mean that the set of variables occurring in F is a subset of $\{X_1, \ldots, X_n\}$. Each formula $F = F(X_1, \ldots, X_n)$ is *associated to a function*

[2] All logarithms in this paper are to the base 2.

$\mathbf{F} = \mathbf{F}(x_1,\dots,x_n) : [0,1]^n \to [0,1]$ as follows:

$$\mathbf{X}_i = \text{the } i\text{th projection } (x_1,\dots,x_n) \mapsto x_i, \tag{2}$$

$$\neg\mathbf{G} = 1 - \mathbf{G}, \tag{3}$$

$$\mathbf{G} \oplus \mathbf{H} = \min(1, \mathbf{G} + \mathbf{H}), \tag{4}$$

$$\mathbf{G} \odot \mathbf{H} = \max(0, \mathbf{G} + \mathbf{H} - 1), \tag{5}$$

$$\mathbf{G} \wedge \mathbf{H} = \min(\mathbf{G}, \mathbf{H}), \tag{6}$$

$$\mathbf{G} \vee \mathbf{H} = \max(\mathbf{G}, \mathbf{H}). \tag{7}$$

For any formula $F = F(X_1,\dots,X_n)$ the dependence of $\mathbf{F}$ on the variables $x_1,\dots,x_n$ shall be tacitly understood whenever possible. Two formulas $F(X_1,\dots,X_n)$ and $G(X_1,\dots,X_n)$ are *equivalent*, in symbols, $F \equiv G$, iff $\mathbf{F} = \mathbf{G}$. The following identities show that the lattice operation $\wedge, \vee$ are definable in terms of $\neg, \oplus, \odot$:

$$G \vee H \equiv \neg(\neg G \oplus H) \oplus H \quad \text{and} \quad G \wedge H \equiv \neg(\neg G \odot H) \odot H. \tag{8}$$

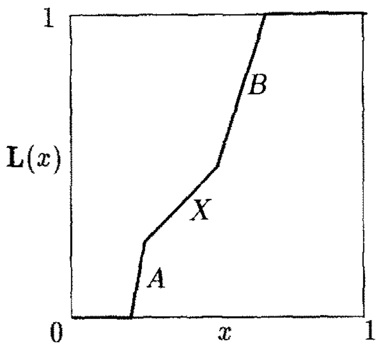
Definition 2.1. Let X be a variable. Then by a *positive literal* in X we mean a negationless formula $P = P(X)$ in which no variable occurs other than X. Stated otherwise, P is built from the variable X by repeated application of the connectives $\odot, \oplus, \wedge, \vee$. Similarly, a *negative literal* in X is built from the negated variable $\neg X$ by repeated application of the connectives $\odot, \oplus, \wedge, \vee$. By a *literal* in X we either mean a positive literal in X, or a negative literal in X. Two literals L and M are said to *have the same sign* iff they are both positive or both negative.

Since by (3)–(7) the De Morgan laws hold for the pair of connectives $(\odot, \oplus)$ as well as for $(\wedge, \vee)$, for each negative literal N one can find in linear time a positive literal P such that $N \equiv \neg P$. And, conversely, for any positive literal P one can quickly find a negative literal N such that $N \equiv \neg P$. To this purpose one just rewrites the formula $\neg P$ in the so called "negation normal form", by pushing the negation symbol inside smaller and smaller subformulas.

The above definition makes the main distinction between our approach to resolution in the infinite-valued calculus, and other approaches mentioned in the introduction. As we shall see in later sections, building on our notion of literal one can naturally introduce significant fragments of the infinite-valued calculus whose satisfiability problems are computable in polynomial time.

Example. Let the positive literal L be given by $L(X) = A(X) \wedge (X \vee B(X))$, where $A = (X \oplus X \oplus X) \odot ((X \oplus X) \odot (X \oplus X \odot X) \oplus X)$ and $B = (X \oplus X \odot X) \odot (X \oplus X)$. Here, for the sake of readability, we are assuming that $\odot$ is more binding than $\oplus$.

Then the graph of the associated function $\mathbf{L}$ is as follows:



Definition 2.2 (*McNaughton* [5]). A map $f:[0,1]^n \to [0,1]$ is called a *McNaughton function* iff it is continuous and there are linear polynomials $p_1,\dots,p_t$ with integer coefficients,

$$p_j(\mathbf{x}) = p_j(x_1,\dots,x_n) = a_{j1}x_1 + \cdots + a_{jn}x_n + b_j$$

such that for each $\mathbf{x}\in[0,1]^n$ there is an index $j=1,\dots,t$ with $f(\mathbf{x}) = p_j(\mathbf{x})$.

Proposition 2.3. *For every formula $F=F(X_1,\dots,X_n)$, its associated function is a McNaughton function $\mathbf{F}:[0,1]^n\to[0,1]$.*

Proof. By induction on the number of connectives occurring in F, as an immediate consequence of (2)–(7). □

Corollary 2.4. *For any positive literal $P=P(X)$, its associated function $\mathbf{P}:[0,1]\to[0,1]$ is monotone increasing ($x\leqslant y$ implies $\mathbf{P}(x)\leqslant\mathbf{P}(y)$), and is not constant – specifically, $\mathbf{P}(0)=0$ and $\mathbf{P}(1)=1$. Moreover, $\mathbf{P}$ is a McNaughton function.*

In Theorem 7.4 we shall prove a converse of the above corollary, thus giving a complete characterization of the expressive power of literals.

Definition 2.5. A *clause* C is a finite set of literals $C=\{L_1,\dots,L_k\}$, such that for each variable X, *C contains at most one positive literal in X, and at most one negative literal in X*. The empty clause is denoted by □. For each clause $C=\{L_1,\dots,L_k\}$ in the variables $X_1,\dots,X_n$, its *associated function* $\mathbf{C}:[0,1]^n\to[0,1]$ is defined by $\mathbf{C}=\mathbf{L}_1\vee\cdots\vee\mathbf{L}_k$. The function associated to the empty clause is the constant 0.

Definition 2.6. Let $\kappa=1,2,\dots,\omega$ be a cardinal. Then any point $\alpha\in[0,1]^\kappa$ is called an *assignment* (or, *model*). Fix an arbitrary real number $0<\theta\leqslant 1$. Then, with reference to (2)–(7), a formula $F=F(X_1,\dots,X_n)$ is said to be *θ-satisfiable* iff $\mathbf{F}(\beta)\geqslant\theta$ for some assignment $\beta\in[0,1]^m$, with $m\geqslant n$. A clause $C=\{L_1,\dots,L_z\}$ is *θ-satisfiable* iff so is the formula $L_1 \vee \cdots \vee L_z$. By definition, the empty clause $\square$ is not θ-satisfied by any assignment.

Notation. Let $\kappa = 1, 2, \ldots, \omega$ be a cardinal. For any assignment $\alpha \in [0,1]^\kappa$ and formula (or clause) G in the variables $X_1, \ldots, X_n$, by writing $\alpha \models_\theta G$ we shall mean that $\kappa \geqslant n$ and $\mathbf{G}(\alpha) \geqslant \theta$. Similarly, by writing $\alpha \not\models_\theta G$ we shall understand that $\kappa \geqslant n$ and $\mathbf{G}(\alpha) < \theta$. Thus in case $\kappa = \omega$, for any assignment $\alpha = (\alpha_1, \alpha_2, \ldots) \in [0,1]^\omega$ and every $G = G(X_1, \ldots, X_n)$ we either have $\alpha \models_\theta G$ or $\alpha \not\models_\theta G$. Further,

$$\alpha \models_\theta G \quad \text{iff} \quad (\alpha_1, \ldots, \alpha_n) \models_\theta G.$$

Trivially, if $0 < m \in \mathbf{Z}$ is not large enough, and $\beta \in [0,1]^m$ is an assignment, it may happen that neither $\beta \models_\theta G$ nor $\beta \not\models_\theta G$. Whenever possible, we shall not get involved in such details about the matching between assignments and variables occurring in formulas.

Definition 2.7. Let Ψ be a set of formulas (resp., a set of clauses). Then Ψ is said to be *θ-satisfiable* iff there is an assignment α such that $\alpha \models_\theta G$ for all $G \in \Psi$. If this is the case we also say that α is a *θ-model* of Ψ, or, equivalently, that Ψ is *θ-satisfied* by α, and we write

$$\alpha \models_\theta \Psi.$$

In particular, for each real number $0 < \theta \leqslant 1$ and assignment α, the empty set of clauses $\emptyset$ is θ-satisfied by α.

Notation. For every formula F, by $\|F\|$ we shall denote the *length* of F, i.e. the number of occurrences of symbols in F. If $\Psi = \{F_1, \ldots, F_n\}$ is a set of formulas, then $\|\Psi\| = \|F_1\| + \cdots + \|F_n\|$. Thus, in particular, for any clause $C = \{L_1, \ldots, L_k\}$, $\|C\| = \|L_1\| + \cdots + \|L_k\|$. Finally, if $\{\Psi_1, \ldots, \Psi_m\}$ is a set of finite sets of formulas then $\|\{\Psi_1, \ldots, \Psi_m\}\| = \|\Psi_1\| + \cdots + \|\Psi_m\|$.

Proposition 2.8. *Every finite set of literals $D = \{M_1, \ldots, M_r\}$ can be transformed into a clause $D^* = \{L_1, \ldots, L_k\}$ such that*

$$M_1 \vee \cdots \vee M_r \equiv L_1 \vee \cdots \vee L_k,$$

by a Turing machine working in a number of steps proportional to the square of the length of D.

Proof. A fast procedure to compute the map $D \mapsto D^*$ is as follows: for each variable X occurring in D, let $P_1(X), \ldots, P_t(X)$ be the list of all positive literals of D in X. Let $N_1(X), \ldots, N_u(X)$ be the list of all negative literals of D in X. If $t \geqslant 2$ then replace all $P_1(X), \ldots, P_t(X)$ by the single literal $P = P_1 \vee \cdots \vee P_t$; similarly, in case $u \geqslant 2$, replace all $N_1(X), \ldots, N_u(X)$ by the single literal $N = N_1 \vee \cdots \vee N_u$. Since the map $D \mapsto D^*$ essentially amounts to sorting the literals of D and then juxtaposing literals of the same sign and variable (with a symbol $\vee$ inserted between any two such literals), this map can be computed in a number of Turing steps proportional to the square of $\|D\|$. (In fact, the map can be computed even faster.) $\square$

Remark. While by (8), the lattice connectives $\wedge$ and $\vee$ are definable in terms of $\oplus$, $\odot$ and $\neg$, this is no longer true when negation is not available (see Proposition 7.5 below). Our initial stipulation (1) about the alphabet Σ, together with Definition 2.1, enables us to say that whenever $L_1,\dots,L_t$ are literals of the same sign, all in the same variable X, then so is $L_1 \vee \cdots \vee L_t$. The above proposition shows that the map $D \mapsto D^*$ does not lead outside polynomial-time computability.

3. Properties of θ-satisfiability

For every set Ψ of formulas we let

$$\mathrm{Mod}_\theta\, \Psi = \{\alpha \in [0,1]^\omega \mid \alpha \models_\theta F \text{ for all } F \in \Psi\}.$$

Proposition 3.1 (Compactness). *Let $0<\theta\leqslant 1$, $\theta\in\mathbf{R}$, Ψ be a set of formulas. If every finite subset $\Phi\subseteq\Psi$ is θ-satisfiable then so is Ψ.*

Proof. We can safely assume that all variables $X|, X||, \dots$ available from our alphabet (1) actually occur in Ψ. By Proposition 2.3, for every finite subset $\Phi\subseteq\Psi$ the set $\mathrm{Mod}_\theta\,\Phi$ is a closed subspace of the Hilbert cube $[0,1]^\omega$. Now the desired conclusion follows from the compactness of $[0,1]^\omega$ with respect to the natural product topology. $\square$

Remark. If instead of the finite alphabet (1) we were given an alphabet with variable symbols X_α, for each ordinal $\alpha<\kappa$ (κ an uncountable cardinal) then, mutatis mutandis, the above result still holds, upon replacing the Hilbert cube $[0,1]^\omega$ by the Tichonoff cube $[0,1]^\kappa$ with the product topology.

Satisfiability and θ-satisfiability. Following [6] we say that a formula $F=F(X_1,\dots,X_n)$ is *satisfiable* iff $\mathbf{F}(\alpha)>0$ for some $\alpha\in[0,1]^n$. A clause $C=\{L_1,\dots,L_m\}$ is *satisfiable* iff the formula $L_1\vee\cdots\vee L_m$ is satisfiable. A set Ψ of formulas is *satisfiable* iff for some assignment α, $\mathbf{F}(\alpha)>0$ for all $F\in\Psi$. Note that compactness does not hold for satisfiability. In [6] the satisfiability problem (SAT_∞) for formulas in the infinite-valued calculus is shown to be NP-complete. The proof is the same whether or not the connectives $\wedge,\vee$ are included in the basic alphabet.

The *generalized satisfiability problem* GENSAT_∞ in the infinite-valued calculus is defined as follows:

INSTANCE: A formula F and a rational number $\theta=c/d$ given by a pair of integers $1\leqslant c\leqslant d$, *in binary notation.*

QUESTION: Is F θ-satisfiable?

The following result shows that GENSAT_∞ is NP-complete.

Theorem 3.2. *$GENSAT_\infty$ is in* NP. *Moreover, there exists a reduction of SAT_∞ to $GENSAT_\infty$ which is computable in polynomial time. Thus $GENSAT_\infty$ is also* NP-*hard.*

Proof. We first show that GENSAT_∞ is in NP. Let $F=F(X_1,\ldots,X_n)$ be a formula. By induction on the number of connectives occurring in F one easily sees that the maximum value $\zeta\in\mathbf{Q}$ of the associated function $\mathbf{F}:[0,1]^n\to[0,1]$ is attained for some assignment $\alpha=(\alpha_1,\ldots,\alpha_n)\in[0,1]^n$, with each α_i being a rational number. Further, the same argument used in the proof of [6, p. 148, Proposition 2.3], shows that the least common denominator d_α of $\alpha_1,\ldots,\alpha_n$ satisfies the inequality

$$d_\alpha\leqslant 2^{(4\|F\|^2)}.$$

(A moment's reflection in the light of (8) shows that it is immaterial whether or not the connectives $\wedge$, $\vee$ are included in the alphabet). As a list of n pairs of relatively prime integers in binary notation, the string α has a length, say, $\leqslant 12\|F\|^2$. By Proposition 2.3 the coefficients of all linear pieces of the McNaughton function $\mathbf{F}$ are integers. Thus, for some integer $k=0,\ldots,d_\alpha$, the maximum value $\zeta=\mathbf{F}(\alpha)$ must coincide with k/d_α. Moreover, k will be nonzero iff F is satisfiable. In particular, F is satisfiable iff it is $(1/d_\alpha)$-satisfiable. In order to compute ζ, writing $\alpha_i=a_i/d_\alpha$, we first replace each variable X_i by α_i. Using (2)–(7), we can compute the value $\mathbf{G}(\alpha)$ of each subformula G of F. Keeping fixed the denominator d_α, we only have to compute no more than $\|F\|$ many additions, subtractions and comparisons between integers $\leqslant d_\alpha$. Therefore, the required maximum value $\zeta=k/d_\alpha=\mathbf{F}(\alpha)$ can be computed in a number of steps bounded by a polynomial in $\|F\|$. Summing up, a fast *nondeterministic* procedure to decide whether F is θ-satisfiable is as follows:

$$\text{guess a short } \theta\text{-satisfying assignment } \alpha=(a_1/d_\alpha,\ldots,a_n/d_\alpha) \text{ for } F, \tag{$\sharp$}$$

$$\text{check that } \mathbf{F}(\alpha)\geqslant\theta=c/d. \tag{$\flat$}$$

We are now in a position to prove NP-hardness. Since, as proved in [6], SAT_∞ is NP-complete, the required polynomial time reduction of instances of SAT_∞ to instances of GENSAT_∞ is immediately obtained upon noting that the binary representation of the integer $2^{(4\|F\|^2)}$ is computable in a number of steps proportional to $\|F\|^2$, and that F is satisfiable iff it is $(2^{-(4\|F\|^2)})$-satisfiable. □

We now define the one-dimensional specialization of Mod_θ for literals:

Definition 3.3. For each literal $L=L(X)$ and real number θ with $0<\theta\leqslant 1$, the *θ-support* of L, $\mathrm{supp}_\theta L$, is defined by

$$\mathrm{supp}_\theta L=\{\alpha\in[0,1]\mid\alpha\models_\theta L\}=\{\alpha\in[0,1]\mid\mathbf{L}(\alpha)\geqslant\theta\}.$$

Note that $\mathrm{supp}_\theta L$ is nonempty. Again, when concerned with computational issues, we shall restrict attention to the case when $0<\theta\leqslant 1$ is *rational.*

Theorem 3.4. *Let $\theta = c/d$ for integers $1 \leqslant c \leqslant d$. Let $P = P(X)$ be a positive literal:*

(i) *Then $\mathrm{supp}_\theta P$ has a smallest element $\mu = \mathrm{minsupp}_\theta P$; μ is a nonzero rational number $\leqslant 1$, whence for uniquely determined relatively prime integers a and b with $1 \leqslant a \leqslant b$ we have*

$$\mu = a/b. \tag{9}$$

Furthermore,

$$a \leqslant b \leqslant d\,\|P\|. \tag{10}$$

(ii) *For some polynomial $\pi = \pi(x, y)$ the binary representation of the pair of integers (a, b) can be computed by a Turing machine working over input (P, c, d) (where the integers c and d are also in binary notation) in a number of steps smaller than $\pi(\log d, \|P\|)$.*

(iii) *Similarly, for any negative literal $N = N(X)$, the set $\mathrm{supp}_\theta N$ has a maximum element $0 \leqslant v = \mathrm{maxsupp}_\theta N < 1$, which is a rational number, and – as a pair of relatively prime integers in binary notation – is computable in a number of steps bounded by a polynomial in $\log d$ and $\|N\|$.*

Proof. (i) We first assume $\theta < 1$. In view of Definition 2.2 together with Corollary 2.4, the McNaughton function $\mathbf{P}$ must be strictly increasing outside the domain where its values are 0 or 1. Thus, there is precisely one point $0 < \alpha \in [0,1]$ such that $\mathbf{P}(\alpha) = \theta$. Since θ is rational and the linear pieces p_i of $\mathbf{P}$ have integer coefficients, it follows that α is rational. Thus, there are uniquely determined relatively prime integers $1 \leqslant a \leqslant b$ satisfying (9). The piecewise linearity of the function $\mathbf{P}$ yields two integers $m \geqslant 1$ and $q \geqslant 0$ such that $\mathbf{P}(x) = mx - q$ for all $x \leqslant \alpha$ sufficiently close to α. Arguing by induction on $\|P\|$, it is easy to see that the linear polynomials $p_i = m_i x - q_i$ representing the linear pieces of $\mathbf{P}$ satisfy the condition

$$m_i, q_i \in \{0, 1, \ldots, \|P\|\}. \tag{11}$$

In particular, $m \leqslant \|P\|$ and $q \leqslant \|P\|$. From $c/d = \theta = \mathbf{P}(\alpha) = ma/b - q = c/d$ we get

$$\frac{a}{b} = \frac{qd + c}{md},$$

whence, $a \leqslant b \leqslant dm \leqslant d\|P\|$, as required to prove (10). The case $\theta = 1$ is similar.

(ii) Again, let us first assume $\theta < 1$. Suppose $\varepsilon \in [0,1]$ to be any point where the function $\mathbf{P}$ is not differentiable. Then in the light of (11) together with Corollary 2.4 we see that ε is rational, say, $\varepsilon = u/e$, for suitable integers $0 \leqslant u \leqslant e$ satisfying the inequality

$$0 < e \leqslant 2\,\|P\|. \tag{12}$$

Instead of a direct attack to the problem of computing the integers a, b, we shall find an interval I containing a/b and such that $\mathbf{P}$ is linear over I. To this purpose, let $B_0 = \{(0,1)(1,1)\}$. Then B_0 is a *basis* in $\mathbf{Z}^2$, (equivalently, B_0 is *unimodular* in the sense that the absolute value of the determinant of the integer matrix whose rows are $(0,1)$ and $(1,1)$ is equal to 1). Let the one-one map $\eta : \mathbf{Z}^2 \to \mathbf{Q}$ transform vectors (p,q), (with $0 \leqslant p \leqslant q$, $q > 0$, $\gcd(p,q) = 1$, where gcd denotes greatest common divisor) into rational numbers $p/q \in [0,1]$. Then using the notation $0 = \frac{0}{1}$, $1 = \frac{1}{1}$, η transforms B_0 into (the endpoints of) the interval $I_0 = [0,1]$. Letting $\alpha_0 = 0$ and $\beta_0 = 1$ be the endpoints of I_0, by assumption we have $0 = \mathbf{P}(\alpha_0) < \theta < \mathbf{P}(\beta_0) = 1$. By taking the *Farey mediant* (sum of numerators divided by sum of denominators) $\mu_1 = (0+1)/(1+1) = \frac{1}{2}$ of these two endpoints $\frac{0}{1}$ and $\frac{1}{1}$, and then applying the inverse map η^{-1} we get two bases in $\mathbf{Z}^2$, namely $B_0' = \{(0,1)(1,2)\}$ and $B_0'' = \{(1,2)(1,1)\}$. As a matter of fact, the unimodularity property is preserved under taking Farey mediants. Further, all fractions obtained by successive application of the mediant operation are automatically in their lowest terms. The above map η now transforms the basis vectors of B_0' and B_0'' into the endpoints of the two intervals, $I_0' = [0,1/2]$ and $I_0'' = [1/2,1]$. In case $\mathbf{P}(1/2) = \theta$ we conclude $\mu = \mathrm{minsupp}_\theta = 1/2$ and we are done. Otherwise, of the two bases B_0' and B_0'', precisely one, denoted B_1, has the following property:

> Letting $I_1 = [\alpha_1, \beta_1]$ be the interval associated with B_1, via the η map, we have the inequalities $\mathbf{P}(\alpha_1) < \theta$ and $\mathbf{P}(\beta_1) > \theta$. Stated otherwise, the graph of the function $\mathbf{P} - \theta$ intersects I_1 in its interior. $(*)$

Proceeding by induction, we obtain a sequence $B_0, B_1, \ldots$ of bases in $\mathbf{Z}^2$, and a decreasing sequence of intervals $I_0 \supseteq I_1 \supseteq \cdots$ with rational endpoints, $I_0 = [\alpha_0, \beta_0], I_1 = [\alpha_1, \beta_1], \ldots$ such that the graph of $\mathbf{P} - \theta$ intersects each I_j in its interior.

Claim. There exists an integer $t \leqslant 2\|P\|$ such that either $\mu_t = a/b$, or the function $\mathbf{P}$ is linear over I_t, and the graph of the function $\mathbf{P} - \theta$ intersects I_t in its interior.

As a matter of fact, it is sufficient to note that the sum of the denominators of the endpoints of the I_j's is strictly increasing with j. By (12), the process of taking mediants and selecting intervals satisfying condition $(*)$ must terminate, and give us an integer $t \leqslant 2\,\|P\|$ satisfying the requirements of the claim.

During each step $i = 1,2,\ldots,t$ we must compute the rational number $\mathbf{P}(\mu_i)$, and in case $\mathbf{P}(\mu_i) - \theta \neq 0$ choose the next basis $B_i \in \{B_{i-1}', B_{i-1}''\}$, and its associated interval I_i, satisfying condition $(*)$, according as whether $\mathbf{P}(\mu_i) > \theta$ or $\mathbf{P}(\mu_i) < \theta$. Writing in binary notation (numerator and denominator of) each mediant μ_i, since the function $\mathbf{P}$ is a composition of at most $\|P\|$ many operations of either form $x + y, \max(x,y), \min(x,y)$ – and none of them increases the denominator of μ_i – it follows from our claim that the rational number $\mathbf{P}(\mu_i)$ can be computed in a number of steps bounded by a polynomial in $\|P\|$ and $\log d$. Thus, by our claim, the computation of the complete list of all intervals $I_0, \ldots, I_t$ can be done in a number of steps bounded by a polynomial in $\|P\|$ and $\log d$. In case some mediant μ_i $(i \leqslant t)$ coincides with a/b we are done. Otherwise, if the graph of $\mathbf{P} - \theta$ intersects each I_i in its interior, knowing that $\mathbf{P}$ coincides with a linear polynomial $mx - q$ over I_t, and having already computed the values of $\mathbf{P}$ at the extremes of I_t, (as two pairs of relatively prime integers in binary notation), it is easy to find the integer coefficients m and q. In particular, for some polynomial π, the binary representation of the pair of integers (a,b) such that $\mathbf{P}(a/b)=c/d$ can be computed in a number of steps bounded by $\pi(\log(d), \|P\|)$, as required.

A similar discussion now settles the case $\theta=1$. Indeed, since the function $\mathbf{P}$ is not differentiable at the point $\mathrm{minsupp}_1 P$, the denominator of $\mathrm{minsupp}_1 P$ is $\leqslant 2\,\|P\|$. As an immediate consequence of (i) and (ii) we get (iii). □

4. Binary and multiple resolution

Let $C=\{L_1,\dots,L_k\}$ be a clause and X a variable occurring in C. Then by Definition 2.5 we have one of the following mutually exclusive cases:
1. X occurs in precisely two literals $L,M\in C$. Then L and M have different signs, and we say that C is *X-saddled.*
2. X occurs in a unique literal $L\in C$. Then according as L is positive or negative we say that C is *X-positive* or C is *X-negative.*

Definition 4.1 (*Binary resolution*). Let $0<\theta\leqslant 1$, $\theta\in\mathbf{R}$, X be a variable, $C_1=\{L_1,\dots,L_p\}$ and $C_2=\{M_1,\dots,M_q\}$ be clauses. Assume that the following conditions hold:
(i) C_1 is X-positive, say, $L_i=L_i(X)$ is the only literal of C_1 in X.
(ii) C_2 is X-negative, say, $M_j=M_j(X)$ is the only literal of C_2 in X.
(iii) $\mathrm{supp}_\theta L_i\cap\mathrm{supp}_\theta M_j=\emptyset$.

Then clauses C_1 and C_2 are said to be *X-resolvable* (with respect to θ). The set of literals $D=(C_1\backslash\{L_i\})\cup(C_2\backslash\{M_j\})$ is turned into a clause

$$D^*=\mathscr{R}(C_1,C_2,X)$$

using the map $D\mapsto D^*$ of Proposition 2.8. D^* is called the *binary X-resolvent* of C_1 and C_2 (with respect to θ). Note that D^* is *X-free*, in the sense that X does not occur in D^*.

The following proposition states that binary resolution is *sound*:

Proposition 4.2. *Suppose C_1 and C_2 are clauses satisfying the above conditions (i)–(iii). Let $D^*=\mathscr{R}(C_1,C_2,X)$. Then for any assignment α such that $\alpha\models_\theta C_1$ and $\alpha\models_\theta C_2$, we automatically have $\alpha\models_\theta D^*$.*

Proof. By (iii) we either have $\alpha\not\models_\theta L_i$ or $\alpha\not\models_\theta M_j$, say without loss of generality, $\alpha\not\models_\theta L_i$. By hypothesis there is a literal $L_r=L_r(Y)\in C_1$ such that $\alpha\models_\theta L_r$. By definition of binary resolvent, either L_r is a literal of D^*, or else there is in D^* a literal $L'(Y)=L_r(Y)\vee Q(Y)$ where $Q(Y)$ has the same sign as L_r. In any case, with the notation of Definition 2.5, it follows that $\mathbf{D}^*(\alpha)\geqslant\mathbf{L}_r(\alpha)\geqslant\theta$. □

We shall soon see that binary resolution is not complete. To define a complete resolution procedure we prepare the following.

Definition 4.3 (*Multiple resolution*). Let $0<\theta\leqslant 1$, $\theta\in\mathbf{R}$. Let X be a variable and $C_1, C_2, \ldots, C_k$ clauses. Suppose the following conditions are satisfied:

(i) C_1 is X-positive, say, $L=L(X)$ is the only literal of C_1 in X;

(ii) C_2 is X-negative, say, $M=M(X)$ is the only literal of C_2 in X;

(iii) $\mathrm{supp}_\theta L \cap \mathrm{supp}_\theta M \neq \emptyset$;

(iv) each of $C_3, \ldots, C_k$ is X-saddled; specifically, for all $t=3,\ldots,k$, let $P_t(X)$, $N_t(X)\in C_t$, with P_t positive and N_t negative;

(v) $\mathrm{supp}_\theta L \cap \mathrm{supp}_\theta M$ is disjoint from

$$(\mathrm{supp}_\theta P_3 \cup \mathrm{supp}_\theta N_3)\cap\cdots\cap(\mathrm{supp}_\theta P_k \cup \mathrm{supp}_\theta N_k).$$

Then the set $\{C_1,\ldots,C_k\}$ is said to be *X-resolvable* (with respect to θ), and by *multiple resolution* we obtain the set of literals

$$D=(C_1\backslash\{L\})\cup(C_2\backslash\{M\})\cup(C_3\backslash\{P_3,N_3\})\cup\cdots\cup(C_k\backslash\{P_k,N_k\})$$

which is transformed into a clause $D^*=\mathscr{R}(C_1,\ldots,C_k,X)$ using the map $D\mapsto D^*$ of Proposition 2.8. We say that D^* is the *multiple X-resolvent* of $C_1,\ldots,C_k$ (with respect to θ).

Again note that the clause $D^*=\mathscr{R}(C_1,\ldots,C_k,X)$ is X-free.

The following proposition states that multiple resolution is sound; its proof immediately follows from the definition, arguing as in the proof of Proposition 4.2:

Proposition 4.4. *Assume the above hypotheses* (i)–(v) *hold. Let α be an assignment such that for all $j=1,\ldots,k$, $\alpha\models_\theta C_j$. Then automatically $\alpha\models_\theta D^*$.*

As we shall see in Theorem 4.6 below, the following generalization of the well known Davis–Putnam procedure is complete for the θ-satisfiability problem in the infinite-valued calculus of Łukasiewicz:

Definition 4.5. Let $0<\theta\leqslant 1$, with $\theta\in\mathbf{R}$. Let $S=\{C_1,\ldots,C_m\}$ be a finite set of clauses, and X a variable occurring in S. Let further

$S_{X\text{-free}}=$ the set of X-free clauses of S.

$S_{X\text{-bin}}=$ the set of binary X-resolvents (with respect to θ) of all possible pairs of resolvable clauses in S.

$S_{X\text{-mult}}=$ the set of multiple X-resolvents (with respect to θ) of all resolvable finite sets of clauses in S.

Let $S'=S_{X\text{-free}}\cup S_{X\text{-bin}}\cup S_{X\text{-mult}}$. Then the set S' is said to be obtained by one *DPP step* (with respect to θ). Note that S' is an X-free set of clauses.

Notation and special assumption. We let $S^{(0)} = S$, $S^{(1)} = S'$, $S^{(2)} = (S^{(1)})', \ldots$. If we further assume θ to be rational, then S' is effectively computable from S. Furthermore, since only finitely many variables $Y_1, \ldots, Y_t$ occur in S, the sequence $S^{(0)}, S^{(1)}, \ldots$ terminates after a number of steps $\leqslant t$. Accordingly, to ensure termination of each DPP step, we shall assume θ to be rational.

The following result follows from an adaptation of the classical resolution techniques to our present definition of literal and clause. We include the proof here for the reader's convenience.

Theorem 4.6 (Completeness). *Let $0 < \theta \leqslant 1$, $\theta \in \mathbf{Q}$. Let W be a set of clauses. Then W is not θ-satisfiable iff for some finite set $S \subseteq W$, and integer $n \geqslant 0$ the empty clause belongs to the set $S^{(n)}$ obtained after n steps of DPP (with respect to θ).*

Proof. The ($\Leftarrow$) direction follows from the soundness of resolution, Proposition 4.4. Concerning the ($\Rightarrow$) direction, by compactness (Proposition 3.1) some finite set $S \subseteq W$ is not θ-satisfiable. Let $X_1, \ldots, X_n$ be the list of variables occurring in S. It is sufficient to settle the following:

Claim 1. For each $t = 0, \ldots, n-1$, if $S^{(t)}$ is not θ-satisfiable, then $S^{(t+1)}$ is not θ-satisfiable.

To this purpose, let us write

$$S^{(t)} = \{K_1, \ldots, K_q\} \quad \text{and} \quad S^{(t+1)} = \{C_1, \ldots, C_r\}.$$

Let $X_1, \ldots, X_p, X_{p+1} = X$ be the variables occurring in $S^{(t)}$, and $X_1, \ldots, X_p$ be the variables occurring in $S^{(t+1)}$. By way of contradiction assume

$$S^{(t)} \text{ is not } \theta\text{-satisfiable, and} \tag{$**$}$$

$$S^{(t+1)} \text{ is } \theta\text{-satisfiable.} \tag{$***$}$$

Then there is an assignment $\alpha \in [0,1]^p$ such that

$$\alpha \models_\theta C_1 \wedge \cdots \wedge C_r. \tag{13}$$

For each real number $\xi \in [0,1]$ let $\alpha^\xi \in [0,1]^{p+1}$ be the extension of α given by

$$\alpha^\xi_{p+1} = \xi, \quad \alpha^\xi_i = \alpha_i, \quad \text{for each } i = 1, \ldots, p. \tag{14}$$

By ($**$) there is a clause $K^\xi \in S^{(t)}$ such that

$$\alpha^\xi \not\models_\theta K^\xi. \tag{15}$$

In particular, for $\xi = 0$, there is a clause $K^0 \in S^{(t)}$ such that

$$\alpha^0 \not\models_\theta K^0. \tag{16}$$

Trivially, X must occur in K^0 (for otherwise we would have $K^0 \in S^{(t)}_{X\text{-free}} \subseteq S^{(t+1)}$ and $\alpha \not\models_\theta K^0$, contradicting (13)). Furthermore, K^0 is X-positive. For, if K^0 also contains a negative literal $M(X)$, then, in the notation of Definition 2.5, $\mathbf{K}^0(\alpha^0) \geqslant \mathbf{M}(\alpha^0) = 1 \geqslant \theta$, which is impossible. Let $P = P(X)$ be the only literal of K^0 in X. We can safely assume that P satisfies the following *minimality condition*:

Whenever $K \in S^{(t)}$ is an X-positive clause such that $\alpha^0 \not\models_\theta K$ and $Q(X)$ is the only literal in X of K, then $\mathrm{supp}_\theta P \subseteq \mathrm{supp}_\theta Q$.

Similarly, for $\xi = 1$, there is an X-negative clause $K^1 \in S^{(t)}$ such that

$$\alpha^1 \not\models_\theta K^1. \tag{17}$$

Let $N(X)$ be the only literal in X of K^1. As above, we can safely assume that N satisfies the following minimality condition:

Whenever $K \in S^{(t)}$ is an X-negative clause such that $\alpha^1 \not\models_\theta K$ and $R(X)$ is the only literal in X of K, then $\mathrm{supp}_\theta N \subseteq \mathrm{supp}_\theta R$.

Claim 2. $\mathrm{supp}_\theta P \cap \mathrm{supp}_\theta N \neq \emptyset$.

For otherwise (absurdum hypothesis) the binary resolvent

$$D^* = \mathscr{R}(K^0, K^1, X) \in S^{(t+1)}$$

would be a clause in $S^{(t)}_{X\text{-bin}} \subseteq S^{(t+1)}$. Since D^* is X-free, by (***), $\alpha \models_\theta D^*$. Thus, there is an X-free literal $L \in D^*$ such that $\alpha \models_\theta L$. By definition of the map $D \mapsto D^*$ (Proposition 2.8), either $L \in K^0$, or $L \in K^1$, or $L = L_0 \vee L_1$, where the two literals $L_0 \in K^0$ and $L_1 \in K^1$ have the same sign and the same variable. In all possible cases we would contradict either (16) or (17), and Claim 2 is proved.

For suitable real (in fact, rational) numbers $0 < \mu \leqslant \nu < 1$ we can now write

$$\mathrm{supp}_\theta P \cap \mathrm{supp}_\theta N = [\mu, \nu] \neq \emptyset. \tag{18}$$

By the assumed minimality of these θ-supports, whenever T is an X-positive clause such that $\alpha^0 \not\models_\theta T$, U is an X-negative clause such that $\alpha^1 \not\models_\theta U$, and $P'(X) \in T$, $N'(X) \in U$, it follows that

$$\mathrm{supp}_\theta P' \cap \mathrm{supp}_\theta N' \supseteq [\mu, \nu]. \tag{19}$$

Recalling (15), for each $\gamma \in [\mu, \nu]$ let us choose an arbitrary clause $K^\gamma \in S^{(t)}$ such that $\alpha^\gamma \not\models_\theta K^\gamma$.

Claim 3. K^γ is X-saddled.

As a matter of fact, if $Q(X)$ were the only literal in X occurring in K^γ, and, say, Q were positive, then by the above minimality property of P, we would get $\mathrm{supp}_\theta Q \supseteq \mathrm{supp}_\theta P$, whence

$$\mathrm{minsupp}_\theta Q \leqslant \mathrm{minsupp}_\theta P,$$

whence a fortiori, $\mathbf{K}^{\gamma}(\alpha^{\gamma}) \geqslant \mathbf{Q}(\alpha^{\gamma}) \geqslant \theta$, which would contradict (15). Similarly, K^{γ} cannot be X-negative. Since, trivially, X must occur in K^{γ}, Claim 3 is settled.

To conclude the proof of Claim 1, for every real number $\gamma \in [\mu, v]$, let us write

$$K^{\gamma} = \{P^{\gamma}(X), N^{\gamma}(X), \ldots\},$$

with P^{γ} positive and N^{γ} negative. By our assumption about K^{γ} we have the inequalities

$$\theta > \mathbf{K}^{\gamma}(\alpha^{\gamma}) \geqslant \mathbf{P}^{\gamma}(\gamma) \vee \mathbf{N}^{\gamma}(\gamma). \tag{20}$$

By continuity, the inequality $\mathbf{P}^{\gamma}(\gamma) \vee \mathbf{N}^{\gamma}(\gamma) < \theta$ holds in an open interval (a_{γ}, b_{γ}) containing γ. By compactness of the unit real interval, there is a finite set of points $\gamma(1), \ldots, \gamma(z) \in [\mu, v]$ such that

$$(a_{\gamma(1)}, b_{\gamma(1)}) \cup \cdots \cup (a_{\gamma(z)}, b_{\gamma(z)}) \supseteq [\mu, v].$$

Thus, from (18) and (20) it follows that $\mathrm{supp}_{\theta} P \cap \mathrm{supp}_{\theta} N$ is disjoint from

$$(\mathrm{supp}_{\theta} P^{\gamma(1)} \cup \mathrm{supp}_{\theta} N^{\gamma(1)}) \cap \cdots \cap (\mathrm{supp}_{\theta} P^{\gamma(z)} \cup \mathrm{supp}_{\theta} N^{\gamma(z)}).$$

Let the multiple resolvent $E^* \in S^{(t+1)}$ be given by

$$E^* = \mathscr{R}(K^0, K^1, K^{\gamma(1)}, \ldots, K^{\gamma(z)}, X).$$

Since $E^* \in S^{(t)}_{X\text{-mult}} \subseteq S^{(t+1)}$, it follows from (14) together with (***), that $\alpha \models_{\theta} E^*$. Thus, $\alpha \models_{\theta} L$ for some literal $L \in E^*$. Now, L is X-free and either already occurs in some clause $K \in \{K^0, K^1, K^{\gamma(1)}, \ldots, K^{\gamma(z)}\}$, or else, L has the form $L = Q_1 \vee \cdots \vee Q_w$, where $Q_1, \ldots, Q_w$ are literals of the same sign in the same variable $Y \neq X$, and each literal Q_i occurs in some clause $K \in \{K^0, K^1, K^{\gamma(1)}, \ldots, K^{\gamma(z)}\}$. It is safe to assume that $\mathbf{L}(\alpha) = \mathbf{Q}_1(\alpha)$. In case $Q_1 \in K^0$ then by (13) and (14), $\mathbf{K}^0(\alpha^0) \geqslant \mathbf{Q}_1(\alpha) \geqslant \theta$, thus contradicting (16). In case $Q_1 \in K^1$ one similarly obtains a contradiction with (17). In the remaining case, if $Q_1 \in K^{\gamma(r)}$ (for some $r = 1, \ldots, z$) then, again,

$$\mathbf{K}^{\gamma(r)}(\alpha^{\gamma(r)}) \geqslant \mathbf{Q}_1(\alpha^{\gamma(r)}) = \mathbf{L}(\alpha^{\gamma(r)}) = \mathbf{L}(\alpha) \geqslant \theta,$$

thus contradicting (20). One similarly deals with the case when L is already a member of some $K \in \{K^0, K^1, K^{\gamma(1)}, \ldots, K^{\gamma(z)}\}$. Claim 1 is thus settled, and the proof of the theorem is complete. □

Corollary 4.7. *Let S be finite and θ-satisfiable set of clauses in the variables $X_1, \ldots, X_n$, where $0 < \theta \leqslant 1$, $\theta \in \mathbf{Q}$. Let $S^{(0)}, \ldots, S^{(n)}$ be obtained by successive applications of the DPP procedure, and let us assume that for each $t = 1, \ldots, n$, X_t is the variable eliminated in the step $S^{(t-1)} \mapsto S^{(t)}$. Then an assignment $\alpha = (\alpha_{t+1}, \ldots, \alpha_n)$ θ-satisfies $S^{(t)}$ iff for some $\xi \in [0,1]$ the assignment $\alpha^{\xi} = (\xi, \alpha_{t+1}, \ldots, \alpha_n)$ θ-satisfies $S^{(t-1)}$ iff for some $\chi \in [0,1] \cap \mathbf{Q}$ the assignment $\alpha^{\chi} = (\chi, \alpha_{t+1}, \ldots, \alpha_n)$ θ-satisfies $S^{(t-1)}$.*

Proof. Immediate from the proof of the previous theorem. The existence of a rational χ having the desired properties follows from the fact that S is finite, and for each literal L in S the function $\mathbf{L} : [0,1] \to [0,1]$ is a McNaughton function (see Corollary 2.4). $\square$

Example. The following example shows that binary resolution alone, as given by Definition 4.1, is not complete: let $\theta = 1$. Let $\{C_1, C_2, C_3\}$ be the three clauses

$$C_1 = \{X \oplus X\}, \quad C_2 = \{\neg X \oplus \neg X\}, \quad C_3 = \{X \odot X, \neg X \odot \neg X\}.$$

By direct verification we get $\mathrm{supp}_1(X \oplus X) = [1/2, 1]$, $\mathrm{supp}_1(\neg X \oplus \neg X) = [0, 1/2]$, and $\mathrm{supp}_1(X \odot X) \cup \mathrm{supp}_1(\neg X \odot \neg X) = \{0, 1\}$. Thus, on the one hand, the set of clauses $\{C_1, C_2, C_3\}$ is not 1-satisfiable; on the other hand, for each $i, j \in \{1,2,3\}$ with $i \neq j$ no binary resolvent of C_i and C_j exists (with respect to $\theta = 1$). It goes without saying that completeness can be achieved by a suitable relaxation of Definition 4.1, in which the resulting generalized binary X-resolvents need no longer be X-free: however, this would be somewhat against the spirit of classical resolution.

5. Krom clauses (clauses with $\leqslant 2$ literals)

Definition 5.1. We say that a clause C is *Krom* iff it contains at most two literals.

Historically, this terminology is justified by [3], and related references.

The *KROMSAT$_\infty$ problem* is defined as follows:

INSTANCE: a finite set S of Krom clauses, and a rational $0 < \theta = c/d \leqslant 1$, where $1 \leqslant c \leqslant d$ are integers in binary notation.

QUESTION: Is S θ-satisfiable ?

Aim of this section is to prove that KROMSAT$_\infty$ is decidable in deterministic polynomial time, and that models of satisfiable sets of Krom clauses can be constructed in polynomial time, thus generalizing the well known result for the classical propositional calculus.

Given two sets V and W of clauses, suppose that for all assignments $\alpha \in [0,1]^\omega$ we have $\alpha \models_\theta V$ iff $\alpha \models_\theta W$. We then say that V is *θ-equivalent* to W, in symbols

$$V \equiv_\theta W.$$

Since every saddled Krom clause C has the form $C = \{P(X), N(X)\}$ for some variable X, we can naturally generalize Definition 3.3 as follows.

Definition 5.2. The *θ-support* $\mathrm{supp}_\theta C$ of an X-saddled Krom clause C is the union of the θ-supports of its literals. An X-saddled clause C is said to be *θ-trivial* iff its θ-support coincides with $[0,1]$. Two X-saddled Krom clauses $C_1 = \{P_1, N_1\}$ and $C_2 = \{P_2, N_2\}$, are said to form a *θ-critical pair* iff $\mathrm{supp}_\theta C_1 \cup \mathrm{supp}_\theta C_2 \neq [0,1]$. We then let $i \in \{1,2\}$ be equal to 1 if $\mathrm{supp}_\theta P_1 \subseteq \mathrm{supp}_\theta P_2$, and $i = 2$ otherwise. Similarly, let $j \in \{1,2\}$ be equal to 1 if $\mathrm{supp}_\theta N_1 \subseteq \mathrm{supp}_\theta N_2$, and $j = 2$ otherwise. Then the *θ-merger* of the θ-critical pair C_1 and C_2 is the X-saddled Krom clause $\mathscr{M}_\theta(C_1,C_2)=\{P_i,N_j\}$. A set S of Krom clauses is said to be *θ-regular* iff it contains no θ-trivial clauses and no θ-critical pairs.

Note that

$$\mathscr{M}_\theta(C_1,C_2)\equiv_\theta C_1\wedge C_2. \tag{21}$$

Proposition 5.3. *Let $\theta=c/d$, where $c\leqslant d$ are integers $\geqslant 1$. Let S be a finite set of Krom clauses. Then S can be transformed into a θ-regular finite set $S_\theta\equiv_\theta S$, such that all literals occurring in S_θ are among those occurring in S. For some polynomial $\rho=\rho(x,y)$ the map $S\mapsto S_\theta$ is computable in less than $\rho(\log d,\|S\|)$ Turing steps.*

Proof. Using the procedure described in the proof of Theorem 3.4 one first computes, for each X-saddled clause $C=\{P(X),N(X)\}\in S$, the rational numbers $\operatorname{minsupp}_\theta P$ and $\operatorname{maxsupp}_\theta N$ as a pair of integers in binary notation. It is then easy to delete from S all θ-trivial clauses, obtaining a new set S' which is θ-equivalent to S. Indeed, a θ-trivial Krom clause is satisfied by all assignments, and its deletion results in a θ-equivalent set. Suppose now that S' contains two X-saddled Krom clauses C_1, C_2, and let $\operatorname{supp}_\theta C_1=[0,\alpha]\cup[\beta,1]$, and $\operatorname{supp}_\theta C_2=[0,\gamma]\cup[\delta,1]$. Then a moment's reflection shows that C_1 and C_2 form a θ-critical pair iff $\max(\alpha,\gamma)<\min(\beta,\delta)$. By another application of Theorem 3.4, some Turing machine can quickly decide if C_1 and C_2 form a θ-critical pair, and – if this is the case – compute their θ-merger $\mathscr{M}_\theta(C_1,C_2)$ in a number of steps bounded by a polynomial in $\log d$ and $(\|C_1\|+\|C_2\|)$. We then replace C_1 and C_2 by the single clause $\mathscr{M}_\theta(C_1,C_2)$. Thus the number of clauses is reduced by one, and no new literals are introduced by the merging operation. The resulting set of clauses is θ-equivalent to S, by (21). Iterated application of the $\mathscr{M}_\theta$ operation yields in polynomial time the desired conclusion. $\square$

Proposition 5.4. *Let $\theta=c/d$, where $c\leqslant d$ are integers $\geqslant 1$. Let $S=\{C_1,C_2,C_3,\dots,C_k\}$ be a θ-regular set of Krom clauses, where $C_3,\dots,C_k$ are X-saddled, C_1 is X-positive, and C_2 is X-negative. Let $P(X)$ and $N(X)$ be the positive and negative literals of C_1 and C_2, respectively. Assume* $\operatorname{supp}_\theta P\cap\operatorname{supp}_\theta N\neq\emptyset$. *Suppose S is θ-resolvable and let*

$$\mathscr{R}(C_1,C_2,C_3,\dots,C_k,X)$$

be its multiple resolvent. Then there is precisely one clause $C\in\{C_3,\dots,C_k\}$ such that $\mathscr{R}(C_1,C_2,C_3,\dots,C_k,X)\equiv_\theta\mathscr{R}(C_1,C_2,C,X)$. For some polynomial $\psi=\psi(x,y)$, both C and the ternary resolvent *$\mathscr{R}(C_1,C_2,C,X)$ can be found in less than $\psi(d,\|S\|)$ Turing steps.*

Proof. For any X-saddled clause $K\in S$, let us denote by $\overline{\operatorname{supp}}_\theta K$ the open interval $[0,1]\setminus\operatorname{supp}_\theta K$. By hypothesis,

$$\operatorname{supp}_\theta P\cap\operatorname{supp}_\theta N\subseteq\bigcup_{i=3}^{k}\overline{\operatorname{supp}}_\theta C_i.$$

From the θ-regularity assumption for S it follows that the open intervals $\overline{\mathrm{supp}}_\theta\, C_1, \ldots, \overline{\mathrm{supp}}_\theta\, C_k$ are nonzero and pairwise disjoint; thus there is precisely one clause $C \in \{C_3, \ldots, C_k\}$ such that $\mathrm{supp}_\theta P \cap \mathrm{supp}_\theta N \subseteq \overline{\mathrm{supp}}_\theta\, C$. By Theorem 3.4 such C can be computed in a number of Turing steps bounded by a polynomial in $\log d$ and $\|S\|$. By Proposition 2.8, the resolvent $\mathscr{R}(C_1, C_2, C_3, \ldots, C_k, X)$ can also be computed in a number of Turing steps bounded by a polynomial in $\log d$ and $\|S\|$. The equivalence $\mathscr{R}(C_1, C_2, C_3, \ldots, C_k, X) \equiv_\theta \mathscr{R}(C_1, C_2, C, X)$ is immediate by definition of C, upon noting that any X-saddled Krom clause contains no other literals than a positive literal in X and a negative literal in X. $\square$

For our computational purposes, we shall now modify the resolution rule of Definitions 4.1 and 4.3, as follows:

Definition 5.5. Fix a rational number $0 < \theta \leqslant 1$. Let $K_1 = \{P(X), L(Y)\}$ and $K_2 = \{N(X), M(Z)\}$ be two Krom clauses, where K_1 is X-positive, and K_2 is X-negative. Suppose $\mathrm{supp}_\theta P \cap \mathrm{supp}_\theta N = \emptyset$. Then the *modified binary X-resolvent* of K_1 and K_2 (with respect to θ) is the clause $\mathscr{R}^\diamond(K_1, K_2, X)$ given by the following stipulation:
- If L and M are literals in the same variable $Y = Z$ having the same sign, and $Q \in \{L, M\}$ is the literal having the larger θ-support, then $\mathscr{R}^\diamond(K_1, K_2, X) = \{Q\}$.
- Otherwise, $\mathscr{R}^\diamond(K_1, K_2, X) = \mathscr{R}(K_1, K_2, X) = \{L, M\}$.

Given three Krom clauses $C_1 = \{P_1(X), L(Y)\}$, $C_2 = \{N_2(X), M(Z)\}$ and $C_3 = \{P_3(X), N_3(X)\}$ suppose $\mathrm{supp}_\theta P_1 \cap \mathrm{supp}_\theta N_2 \neq \emptyset$, and $\mathrm{supp}_\theta P_1 \cap \mathrm{supp}_\theta N_2 \cap (\mathrm{supp}_\theta P_3 \cup \mathrm{supp}_\theta N_3) = \emptyset$. Then the *modified ternary X-resolvent* of C_1, C_2 and C_3 (with respect to θ) is the clause $\mathscr{R}^\diamond(C_1, C_2, C_3, X)$ given by the following stipulation:
- If L and M are literals in the same variable $Y = Z$ having the same sign, and $Q \in \{L, M\}$ is the literal having the larger θ-support, then $\mathscr{R}^\diamond(C_1, C_2, C_3, X) = \{Q\}$.
- Otherwise, $\mathscr{R}^\diamond(C_1, C_2, C_3, X) = \mathscr{R}(C_1, C_2, C_3, X) = \{L, M\}$.

Proposition 5.6. *The modified binary and ternary resolution of Definition 5.5 is sound and complete for the θ-satisfiability problem of θ-regular sets of Krom clauses.*

Proof. One first observes that for each $k \in \{2, 3\}$, if the clauses $C_1, \ldots, C_k$ are X-resolvable then $\mathscr{R}(C_1, \ldots, C_k, X) \equiv_\theta \mathscr{R}^\diamond(C_1, \ldots, C_k, X)$. The desired result now easily follows from soundness and completeness of resolution, together with Proposition 5.4. $\square$

We now define the correspondingly modified DPP procedure:

Definition 5.7. Let $0 < \theta \leqslant 1$, $\theta \in \mathbf{Q}$. Let S be a θ-regular finite set of Krom clauses. Let X be a variable occurring in S. Let, as in Definition 4.5, $S_{X\text{-free}} =$ the set of all X-free clauses in S. Then, without explicitly mentioning the dependence on θ, we let

$S_{X\text{-modbin}} =$ the set of modified binary X-resolvents of clauses in S.

$S_{X\text{-modter}} =$ the set of modified ternary X-resolvents of clauses in S.

In the light of Proposition 5.3 let $S' = (S_{X\text{-free}} \cup S_{X\text{-modbin}} \cup S_{X\text{-modter}})_\theta$. Then the set S' is said to be obtained from S by one *modified DPP step* (*taken on the variable* X). In case no variable occurs in S, then S coincides with the empty set of clauses $\emptyset$, or with the singleton set $\{\Box\}$. In either case we stipulate that $S' = S$.

For any fixed enumeration $Z_1, \ldots, Z_n$ of the variables occurring in S we shall adopt the following notation:

$$S^{[0]} = S$$

and, for all integers $1 \leqslant t \leqslant n$,

$$S^{[t]} = (S^{[t-1]})',$$

where the step leading from $S^{[t-1]}$ to $S^{[t]}$ is tacitly assumed to be taken on the variable Z_t.

Lemma 5.8. *Let as above, $0 < \theta = c/d \leqslant 1$ be a rational number, for integers $1 \leqslant c \leqslant d$ in binary notation. Let S be a finite θ-regular set of Krom clauses. Let $X_1, \ldots, X_n$ be the variables occurring in S. Let $S^{[0]}, S^{[1]}, \ldots$ be the sequence of θ-regular sets of Krom clauses obtained by repeated application of the modified DPP procedure, according to some fixed but otherwise arbitrary enumeration of the variables.*

(i) *Then one of the following mutually exclusive cases must occur:*
  - *either $S^{[n]} = \emptyset$ (in which case S is θ-satisfiable),*
  - *or $S^{[n]} = \{\Box\}$ (whence S is not θ-satisfiable).*

(ii) *Further, for some polynomial $\mu = \mu(x, y)$, the sequence $S^{[0]}, \ldots, S^{[n]}$ is computable in less than $\mu(\log d, \|S\|)$ steps by a Turing machine working over input (S, c, d).*

Proof. (i) This follows from Proposition 5.6, noting that each modified DPP step eliminates a variable without introducing new ones.

(ii) Let $\{L_1, \ldots, L_l\}$ be the set of literals in S. For each $j = 1, \ldots, l$ let the rational number σ_j be defined by: $\sigma_j = \operatorname{minsupp}_\theta L_j$ if L_j is positive, and $\sigma_j = \operatorname{maxsupp}_\theta L_j$, if L_j is negative. By Theorem 3.4 the list

$$(L_1, \sigma_1), \ldots, (L_l, \sigma_l)$$

can be computed in a number of Turing steps bounded by a polynomial in $\log d$ and $\|S\|$. Let $K(L_1, \ldots, L_l)$ be the set of all Krom clauses obtainable from the literals in S. Then the number k of elements of $K(L_1, \ldots, L_l)$ satisfies the inequality

$$k \leqslant 4l^2 + 2l + 1 \leqslant 4\|S\|^2 + 2\|S\| + 1. \tag{22}$$

By definition, neither the $\mathscr{M}_\theta$ operation, nor the modified resolution operation $\mathscr{R}^\diamond$ generate new literals. It follows that all literals occurring in $S^{[0]} \cup \cdots \cup S^{[n]}$ will be among $\{L_1, \ldots, L_l\}$. This key observation allows us to give a fast procedure to compute the sequence $S^{[0]}, \ldots, S^{[n]}$ as follows: Focusing attention on the ith DPP step $S^{[i]} \mapsto S^{[i+1]}$, avoiding all trivialities, and assuming $S^{[i]} \neq \emptyset$ and $S^{[i]} \neq \{\Box\}$, there is a unique variable

X such that $S^{[i+1]}$ is obtained by computing modified binary and ternary X-resolvents of clauses in $S^{[i]}$.

Claim. There is a polynomial $v=v(x,y)$ such that, for each $i=0,\dots,n-1$, $S^{[i+1]}$ can be computed by a Turing machine over input $(S^{[i]},c,d)$ in less than $v(\log d,\|S\|)$ steps.

Indeed, the computation of $S^{[i]}_{X\text{-modbin}}$ (resp., of $S^{[i]}_{X\text{-modter}}$) requires examination of the θ-supports of at most k^2 pairs (resp., at most k^3 triplets) of clauses in $S^{[i]}$. Using the above list $(L_1,\sigma_1),\dots,(L_l,\sigma_l)$, the set

$$S^{[i]}_{X\text{-free}}\cup S^{[i]}_{X\text{-modbin}}\cup S^{[i]}_{X\text{-modter}}$$

can be computed in a number of Turing steps bounded by a fixed polynomial in $\log d$ and $\|S\|$, independently of i. An application of Proposition 5.3, together with (22) shows that each $S^{[i+1]}$ is computable by a Turing machine over input $(S^{[i]},c,d)$ in a number of steps bounded by a *fixed* polynomial in $\log d$ and $\|S\|$. It is now easy to find a polynomial v satisfying the requirements of our claim. To conclude the proof one simply notes that the number n of DPP steps under consideration is $\leqslant\|S\|$. □

Theorem 5.9. *There is a polynomial χ in two variables, and a Turing machine $\mathscr{K}$ such that for every rational $0<\theta=c/d\leqslant 1$ (with integers $1\leqslant c\leqslant d$ in binary notation) and every finite set S of Krom clauses, over input (S,c,d), $\mathscr{K}$ decides θ-satisfiability of S in less than $\chi(\log d,\|S\|)$ steps. Thus, the KROMSAT$_\infty$ problem is decidable in polynomial time.*

Proof. Immediate from the above lemma together with Proposition 5.3. □

In the rest of this section we shall describe a fast procedure to build models for θ-satisfiable finite sets of Krom clauses.

Proposition 5.10. *Let $0<\theta=c/d\leqslant 1$, where $c\leqslant d$ are positive integers in binary notation. Let $S=\{C_1,\dots,C_z\}$ be a θ-satisfiable finite set of Krom clauses in which no other variable than X occurs. Let $\mathbf{S}=\mathbf{C}_1\wedge\cdots\wedge\mathbf{C}_z$, be the McNaughton function associated to S, where each $\mathbf{C}_r$ is as in Definition 2.5. Then a rational number $\xi=a/b\in[0,1]$ such that $\mathbf{S}(\xi)\geqslant\theta$ (with relatively prime integers $1\leqslant a\leqslant b$ in binary notation) can be obtained in a number of Turing steps bounded by a polynomial in $\log d$ and $\|S\|$.*

Proof. In the light of Proposition 5.3, we can safely assume S to be θ-regular. Let S^+ be the set of X-positive clauses of S. Let S^- be the set of X-negative clauses of S. If S^+ is empty, then $\xi=0$ will be a rational θ-model of S. Otherwise, if S^- is empty, then $\xi=1$ will be a rational θ-model of S. If both S^+ and S^- are nonempty, let the rational numbers μ and v be defined by $\mu=\max\{\min\operatorname{supp}_\theta P\mid P\in S^+\}$ and

$\nu = \min\{\max \operatorname{supp}_\theta N \mid N \in S^-\}$. Since S is θ-satisfiable, we can write $0<\mu\leqslant\nu<1$. Recalling the notation of Proposition 5.4 we argue by cases:

Case 1: There is no X-saddled clause $C \in S$ such that $\mu \in \overline{\operatorname{supp}_\theta} C$. Then $\xi = \mu$ will be a rational θ-model of S.

Case 2: There is an X-saddled clause $C = \{P(X), N(X)\} \in S$ such that $\mu \in \overline{\operatorname{supp}_\theta} C$. Then, since S is θ-satisfiable, the rational $\xi = \min \operatorname{supp}_\theta P$ must satisfy the inequality $\xi \leqslant \nu$, whence, by the assumed θ-regularity of S, ξ is a θ-model of S.

In either case, by Theorem 3.4, the computation of a pair of relatively prime integers in binary notation representing the value of ξ can be done in a number of steps bounded by a polynomial in $\log d$ and $\|S\|$. One promptly decides whether S^+ or S^- is empty (whence ξ can be set equal to 0 or 1, respectively), or else, whether the value of ξ should be computed in accordance with Case 1 or 2. □

Corollary 5.11. *There is a polynomial $\tau = \tau(x, y)$ and a Turing machine $\mathscr{T}$ such that, given any rational $0<\theta=c/d\leqslant 1$ (where $1\leqslant c\leqslant d$ are integers in binary notation) and any θ-satisfiable finite set S of Krom clauses in the variables $X_1,\dots,X_n$, over input (S,c,d) machine $\mathscr{T}$ outputs, in less than $\tau(\log d, \|S\|)$ steps, a θ-model α of S, say, $\alpha=(\alpha_1,\dots,\alpha_n)$, where each α_i is a pair of relatively prime integers in binary notation.*

Proof. By Proposition 5.3 we can safely restrict to the case when $S = S^{[0]}$ is θ-regular. Let $S^{[0]},\dots,S^{[n]}=\emptyset$ be the sets of Krom clauses obtained by the procedure of Lemma 5.8. Without loss of generality, we may assume that $S^{[i]}$ only contains the variables $X_{i+1},\dots,X_n$. There is a polynomial $\mu=\mu(x,y)$ such that the list

$$S^{[n]}, S^{[n-1]},\dots, S^{[0]}$$

is computable in less than $\mu(\log d, \|S\|)$ steps. Further, by Proposition 5.6, together with Corollary 4.7, every $S^{[i]}$ is θ-satisfiable. Thus in particular, skipping all trivialities, and using Proposition 5.10, we obtain a θ-model $\alpha_n \in [0,1]\cap\mathbf{Q}$ for $S^{[n-1]}$, in a number of steps bounded by a polynomial in $\log d$ and $\|S\|$.

Proceeding by induction, let $1\leqslant i\leqslant n-1$, and assume

$$\alpha = (\alpha_{i+1},\dots,\alpha_n) \in ([0,1]\cap\mathbf{Q})^{n-i}$$

to be a θ-model of $\mathbf{S}^{[i]}$, where, as in Proposition 5.10, $\mathbf{S}^{[i]}$ denotes the McNaughton function associated to the set of clauses $S^{[i]}$. For some rational $\alpha_i \in [0,1]$, Corollary 4.7 yields an extension $\alpha' = (\alpha_i,\alpha_{i+1},\dots,\alpha_n)$ such that $\alpha' \models_\theta \mathbf{S}^{[i-1]}$.

There only remains to evaluate the complexity of this model building procedure. To this purpose, let us define $S^* \subseteq S^{[i-1]}$ by the following stipulation: A clause $C \in S^{[i-1]}$ belongs to S^* iff for some index $j=i+1,\dots,n$, C contains a literal $L=L(X_j)$ such that $\alpha_j \models_\theta L$. Note that all X_i-free clauses of $S^{[i-1]}$ must belong to S^*. Let $S^{**}=S^{[i-1]}\setminus S^*$. Let the set $W^{[i-1]}$ of clauses be obtained by deleting from each clause of S^{**} every literal $L=L(X_t)$, whenever $t=i+1,\dots,n$. By construction, $W^{[i-1]}$ is a set of Krom clauses only containing the variable X_i. Again by Lemma 5.8, $W^{[i-1]}$ is computable in a number of steps bounded by a polynomial in $\log d$ and $\|S\|$, independent of i.

A moment's reflection, recalling the proof of Theorem 4.6, shows that for every $\xi \in [0,1]$,

$$\mathbf{S}^{[i-1]}(\xi, \alpha_{i+1}, \ldots, \alpha_n) \geqslant \theta \quad \text{iff} \quad \mathbf{W}^{[i-1]}(\xi) \geqslant \theta.$$

By induction hypothesis, together with Corollary 4.7, the set $S^{[i-1]}$ is θ-satisfiable by some assignment of the form $(\xi, \alpha_{i+1}, \ldots, \alpha_n)$, (where $\xi \in [0,1] \cap \mathbf{Q}$). From Proposition 5.10 it follows that the binary representation of the pair (a,b) of relatively prime integers $1 \leqslant a \leqslant b$ such that $\alpha_i = a/b \models_\theta W^{[i-1]}$ is obtainable in a number of Turing steps bounded by a polynomial in $\log d$ and $\|W^{[i-1]}\|$, independently of i. Therefore, by our previous discussion, the binary representation of the pair (a,b) is computable in a number of Turing steps bounded by a polynomial in $\log d$ and $\|S\|$. In conclusion, our inductive computation of

$$(\alpha_{n-1}, \alpha_n) \models_\theta S^{[n-2]}, \quad (\alpha_{n-2}, \alpha_{n-1}, \alpha_n) \models_\theta S^{[n-3]}, \ldots, (\alpha_1, \ldots, \alpha_n) \models_\theta S^{[0]}$$

can be performed in a number of steps bounded by a fixed polynomial in $\log d$ and $\|S\|$. This completes the proof. □

Remark. (1) The above method can be applied, mutatis mutandis, to any finite set of (possibly, non Krom) clauses, yielding an effective model building procedure. However, since our analysis in Lemma 5.8 does not hold in general, by analogy with the classical case, the number $\max(\|S^{[0]}\|, \ldots, \|S^{[n]}\|)$ often grows exponentially with respect to $\|S\|$. As a consequence, the above inductive computation of $(\alpha_{n-1}, \alpha_n) \models_\theta S^{[n-2]}, (\alpha_{n-2}, \alpha_{n-1}, \alpha_n) \models_\theta S^{[n-3]}, \ldots, (\alpha_1, \ldots, \alpha_n) \models_\theta S^{[0]}$ requires an exponential number of Turing steps.

(2) As a consequence of Corollary 5.11, for any finite and satisfiable set S of Krom clauses one can effectively compute the maximum value $\theta \in \mathbf{Q}$ such that there exists $\beta \models_\theta S$. Using now a routine binary search procedure, for any integer k such that $2^{-k} < \theta$ one can find an assignment α such that $\alpha \models_{\theta - 2^{-k}} S$, in a number of steps bounded by a polynomial in k and $\|S\|$.

6. Horn clauses and least model computation

In this section we shall concentrate on model building for satisfiable sets of Horn clauses. Rather than developing ad hoc resolution techniques for Horn clauses, we shall generalize to the infinite-valued calculus of Łukasiewicz the traditional fixpoint constructions of classical propositional logic. Indeed, these techniques have far reaching generalizations. For the reader's convenience, we shall assume no prerequisite on this subject.

Definition 6.1. A clause K is a *Horn clause* iff it has at most one positive literal. A Horn clause K is said to be *nonnegative* iff is contains precisely one positive literal. A nonnegative Horn clause K containing precisely one literal is called a *unit*. Thus, K consists of a single positive literal.

The $HORNSAT_\infty$ problem is defined as follows:

INSTANCE: A finite set S of Horn clauses, and a rational $0<\theta=c/d\leqslant 1$, where $1\leqslant c\leqslant d$ are integers in binary notation.

QUESTION: Is S θ-satisfiable ?

Aim of this section is to establish that HORNSAT$_\infty$ is decidable in polynomial time, thus generalizing the well known result for the classical propositional calculus. We can safely restrict attention to assignments $\alpha=(\alpha_1,\alpha_2,\ldots)\in[0,1]^\omega$.

Definition 6.2. The *natural lattice-order* $\preccurlyeq$ over $[0,1]^\omega$ is defined by stipulating that for any two assignments $\alpha,\beta\in[0,1]^\omega$, $\alpha\preccurlyeq\beta$ iff for each $i=1,2,\ldots$, $\alpha_i\leqslant\beta_i$. The infimum $\beta\wedge\gamma$ of two assignments $\beta,\gamma\in[0,1]^\omega$ is given by $(\beta\wedge\gamma)_i=\min(\beta_i,\gamma_i)$, $i=1,2,\ldots$. We write $\alpha\prec\beta$ iff $\alpha\preccurlyeq\beta$, and $\beta\neq\alpha$.

Proposition 6.3. *Let* $0<\theta\leqslant 1$, $\theta\in\mathbf{R}$. *Let S be a set of Horn clauses.*

(i) *For any two assignments $\alpha,\beta\in[0,1]^\omega$ if $\alpha\models_\theta S$ and $\beta\models_\theta S$ then $\alpha\wedge\beta\models_\theta S$.*

(ii) *If S is θ-satisfiable, then the set*

$$\mathrm{Mod}_\theta S=\{\alpha\in[0,1]^\omega\mid \text{for all } K\in S,\ \alpha\models_\theta K\}$$

has a $\preccurlyeq$-least element, denoted $\min(\mathrm{Mod}_\theta S)$.

Proof. (i) It is sufficient to consider a single Horn clause $K=\{L_1,\ldots,L_t\}$. If K does not contain any positive literals the result follows from the fact that negative literals are *antimonotone*, i.e., their associated functions are monotonically decreasing. Otherwise, letting without loss of generality L_t be the only positive literal of K, if both $\alpha\models_\theta L_t$ and $\beta\models_\theta L_t$, then $\alpha\wedge\beta\models_\theta L_t$ and we are done. There only remains to consider the case when for some $j=1,\ldots,t-1$, either $\alpha\models_\theta L_j$ or $\beta\models_\theta L_j$. Then, by the above antimonotonicity property, $\alpha\wedge\beta\models_\theta L_j$, as required to complete the proof of (i).

(ii) Since $\mathrm{Mod}_\theta S$ is nonempty, for each $i=1,2,\ldots$ let

$$(\mathrm{Mod}_\theta S)_i=\{\xi\in[0,1]\mid \alpha_i=\xi,\ \text{for some } \alpha\in\mathrm{Mod}_\theta S\}$$

be the projection of $\mathrm{Mod}_\theta S$ into the ith coordinate axis; further, let the real number $\mu_i\in[0,1]$ be defined by $\mu_i=\inf(\mathrm{Mod}_\theta S)_i$, and let the assignment $\mu\in[0,1]^\omega$ be given by $\mu=(\mu_1,\mu_2,\ldots)$.

Claim. $\mu\in\mathrm{Mod}_\theta S$.

Otherwise (absurdum hypothesis), let $K=\{L_1,\ldots,L_m\}\in S$ be such that $\mu\not\models_\theta K$. By assumption, whenever $\beta\in\mathrm{Mod}_\theta S\subseteq\mathrm{Mod}_\theta K$, then $\mu\preccurlyeq\beta$. Since negative literals have the antimonotonicity property, the only possibility for μ not to θ-satisfy K is that K contains a positive literal, say $P=P(X)=L_m$. For some index $j=1,2,\ldots$, the variable X coincides with the jth variable X_j. Regarding the function $\mathbf{P}$ as only dependent on the single variable x_j, from our absurdum hypothesis it follows that $\mathbf{P}(\mu_j) < \theta$. By continuity, for all suitably small real numbers $\varepsilon > 0$ we have $\mathbf{P}(\mu_j + \varepsilon) < \theta$. By definition of μ, ε can be chosen in such a way that there is $\alpha \in \mathrm{Mod}_\theta S$ with $\alpha_j = \mu_j + \varepsilon$. Upon noting that $\alpha_i \geqslant \mu_i$ for all $i = 1, 2, \ldots$, from the antimonotonicity property of the literals $L_1, \ldots, L_{m-1}$ it follows that $\mathbf{K}(\alpha) < \theta$, i.e., $\alpha \not\models_\theta K$, thus contradicting our assumption about α. Our claim is settled, and the proof of (ii) is completed. $\square$

Proposition 6.4. *Let* $0 < \theta \leqslant 1$, $\theta \in \mathbf{R}$. *Let* S *be a set of Horn clauses. Let* $S^+ \subseteq S$ *be the set of nonnegative clauses in* S. *Then* S^+ *is* θ*-satisfiable. Further, if* S *is* θ*-satisfiable then* $\min(\mathrm{Mod}_\theta S) = \min(\mathrm{Mod}_\theta S^+)$.

Proof. Trivially, the assignment $(1, 1, \ldots) \in [0, 1]^\omega$ θ-satisfies each clause in S^+. To conclude the proof, let $S^- = S \backslash S^+$, and let $\alpha \in [0, 1]^\omega$ be an arbitrary assignment such that $\alpha \models_\theta S$. Let $\mu = \min(\mathrm{Mod}_\theta S^+)$ be as given by Proposition 6.3(ii). Since $\alpha \models_\theta S^+$, from Proposition 6.3(i) it follows that $\mu \wedge \alpha \models_\theta S^+$, whence $\mu \leqslant \alpha$. By the antimonotonicity properties of all literals in S^- we conclude that $\mu \models_\theta S$, whence $\mu = \min(\mathrm{Mod}_\theta S)$. $\square$

Notation. Every nonnegative Horn clause C can be written as

$$C = P_1 \wedge \cdots \wedge P_n \Rightarrow P \quad (n \geqslant 0), \tag{23}$$

where $P_1, \ldots, P_n$, P are positive literals, and $A \Rightarrow B$ is an abbreviation of $\neg A \vee B$. (*Warning*: This is not Łukasiewicz's implication.) Following a classical tradition we also write

$$C_{\mathrm{Body}} = P_1 \wedge \cdots \wedge P_n \quad \text{and} \quad C_{\mathrm{Head}} = P.$$

This allows us to concentrate only on positive literals. There are some obvious adjustments in our previous stipulations about associated functions. Thus for instance, in case C is a unit, $C_{\mathrm{Body}} = \emptyset$; then the associated function $\mathbf{C}_{\mathrm{Body}}$ is identified with the constant function 1.

As an immediate consequence of the definition we have

Proposition 6.5. *Fix a real number* θ *with* $0 < \theta \leqslant 1$, *and a nonnegative Horn clause* C. *Then for every* $\alpha \in [0, 1]^\omega$ *the following conditions are equivalent*:
(i) $\alpha \models_\theta C$,
(ii) *If* $\mathbf{C}_{\mathrm{Body}}(\alpha) > 1 - \theta$ *then* $\mathbf{C}_{\mathrm{Head}}(\alpha) \geqslant \theta$.

Definition 6.6. For every (possibly infinite) set S of nonnegative Horn clauses, variable X, real number $0 < \theta \leqslant 1$, and assignment $\alpha \in [0, 1]^\omega$, the set $A = \mathscr{A}(S, X, \theta, \alpha)$ of positive literals in X is defined by

$$A = \{P = P(X) \mid \text{for some } C \in S,\ C_{\mathrm{Head}} = P \text{ and } \mathbf{C}_{\mathrm{Body}}(\alpha) > 1 - \theta\}.$$

Further, the operator $T_S^\theta : [0,1]^\omega \to [0,1]^\omega$ is defined by stipulating that, for every index $i = 1,2,3,\ldots$ and assignment $\alpha \in [0,1]^\omega$

$$(T_S^\theta(\alpha))_i = \begin{cases} 0, & \text{if } \mathscr{A}(S,X_i,\theta,\alpha) = \emptyset \\ \sup\{\text{minsupp}_\theta P \mid P \in \mathscr{A}(S,X_i,\theta,\alpha)\}, & \text{otherwise.} \end{cases}$$

Proposition 6.7. *Let $0<\theta\leqslant 1$, $\theta \in \mathbf{R}$. Let S be a set of nonnegative Horn clauses. Assume further $\alpha,\beta \in [0,1]^\omega$, and $\alpha \leqslant \beta$. Then we have*
(i) *For every variable X, $\mathscr{A}(S,X,\theta,\alpha) \subseteq \mathscr{A}(S,X,\theta,\beta)$.*
(ii) *T_S^θ is* monotone, *in the sense that $T_S^\theta(\alpha) \leqslant T_S^\theta(\beta)$.*

Proof. (i) If $P \in \mathscr{A}(S,X,\theta,\alpha)$ then let $K \in S$ be such that $K_{\text{Head}} = P$ and $\mathbf{K}_{\text{Body}}(\alpha) > 1-\theta$. Since $\mathbf{K}_{\text{Body}}$ is monotonically increasing, $\mathbf{K}_{\text{Body}}(\beta) > 1-\theta$, whence $P \in \mathscr{A}(S,X,\theta,\beta)$. The proof of (ii) follows now immediately. □

Proposition 6.8. *Let $0<\theta\leqslant 1$, $\theta \in \mathbf{R}$. Let S be a set of nonnegative Horn clauses. Then for every assignment $\alpha \in [0,1]^\omega$ we have*

$$\alpha \models_\theta S \quad \text{iff} \quad T_S^\theta(\alpha) \leqslant \alpha.$$

Proof. ($\Leftarrow$) Assume $T_S^\theta(\alpha) \leqslant \alpha$, and let $K \in S$ be such that $\mathbf{K}_{\text{Body}}(\alpha) > 1-\theta$ and $K_{\text{Head}} = P = P(X_i)$, for some index $i = 1,2,\ldots$. Then $P \in \mathscr{A}(S,X_i,\theta,\alpha)$. By hypothesis we get $\alpha_i \geqslant (T_S^\theta(\alpha))_i \geqslant \text{minsupp}_\theta P$, whence $\mathbf{P}(\alpha) \geqslant \theta$. By Proposition 6.5, $\alpha \models_\theta K$.

($\Rightarrow$) Let $\alpha \models_\theta S$, with the intent of proving $(T_S^\theta(\alpha))_i \leqslant \alpha_i$ for all $i = 1,2,\ldots$. Let $X = X_i$. If $\mathscr{A}(S,X,\theta,\alpha) = \emptyset$, we are done. Otherwise, let $P \in \mathscr{A}(S,X,\theta,\alpha)$ be the head of a clause $K \in S$ such that $\mathbf{K}_{\text{Body}}(\alpha) > 1-\theta$. By hypothesis, together with Proposition 6.5, $\mathbf{P}(\alpha) \geqslant \theta$, and $\alpha_i \geqslant \text{minsupp}_\theta P$. Since P is arbitrary in $\mathscr{A}(S,X,\theta,\alpha)$ we conclude that $\alpha_i \geqslant (T_S^\theta(\alpha))_i$, as required. □

Proposition 6.9. *Let $0<\theta\leqslant 1$, $\theta \in \mathbf{R}$. Let S be a set of nonnegative Horn clauses. Then T_S^θ is* (monotone) σ-complete, *in the following sense*: *Whenever $\alpha^1 \leqslant \alpha^2 \leqslant \cdots \leqslant \alpha^n \leqslant \cdots$ is a sequence of assignments in $[0,1]^\omega$, and*

$$\alpha = \bigsqcup_n \alpha^n$$

denotes their least upper bound (with respect to the lattice-order $\leqslant$), then we have

$$T_S^\theta(\alpha) = \bigsqcup_n T_S^\theta(\alpha^n).$$

Proof. Note that α is well defined, and, specifically, $\alpha_i = \sup_n \alpha_i^n$. Similarly, $\bigsqcup_n T_S^\theta(\alpha^n)$ is well defined. Fix an index $i = 1,2,3,\ldots,$ and write $A_i = \mathscr{A}(S,X_i,\theta,\alpha)$ and $A_i^n = \mathscr{A}(S,X_i,\theta,\alpha^n)$. By Proposition 6.7 we have $A_i^1 \subseteq A_i^2 \subseteq \cdots$, whence $(T_S^\theta(\alpha^1))_i \leqslant (T_S^\theta(\alpha^2))_i \leqslant \cdots$.

Claim. $A_i = \bigcup_n A_i^n$.

Indeed, if $P = P(X_i) \in A_i^m$ and $P = K_{\text{Head}}$ for some $K \in S$ such that $\mathbf{K}_{\text{Body}}(\alpha^m) > 1 - \theta$, then by monotonicity (Proposition 6.7), $\mathbf{K}_{\text{Body}}(\alpha) > 1 - \theta$, whence $P \in A_i$.

Conversely, if $P \in A_i$, assume $P = C_{\text{Head}}$ for some clause $C \in S$ such that $\mathbf{C}_{\text{Body}}(\alpha) > 1 - \theta$. The function $\mathbf{C}_{\text{Body}}$ depends only on finitely many variables, and is continuous and monotone increasing in each variable. Since by hypothesis, for each $j = 1, 2, 3, \ldots$, $\lim_{n\to\infty} \alpha_j^n = \alpha_j$, there exists an integer $1 \leqslant m$ such that $\mathbf{C}_{\text{Body}}(\alpha^m) > 1 - \theta$. Therefore, $P \in \bigcup_n A_i^n$, as required to settle our claim.

Now, to conclude the proof, let us write

$$\begin{aligned}(T_S^\theta(\alpha))_i &= \sup\{\text{minsupp}_\theta P \mid P \in A_i\}\\ &= \sup\left\{\text{minsupp}_\theta P \mid P \in \bigcup_n A_i^n\right\}\\ &= \sup_m\{\text{minsupp}_\theta P \mid P \in A_i^m\}\\ &= \sup_m (T_S^\theta(\alpha^m))_i,\end{aligned}$$

whence the desired conclusion follows by definition of $\bigsqcup$. □

Definition 6.10. Let $0 < \theta \leqslant 1$ with $\theta \in \mathbf{R}$. Let S be an arbitrary set of nonnegative Horn clauses. Then the sequence of assignments $\alpha^0, \alpha^1, \alpha^2, \ldots$ ($\alpha^i \in [0,1]^\omega$) is inductively defined by

$$\alpha^0 = (0, 0, 0, \ldots),$$

$$\alpha^{n+1} = T_S^\theta(\alpha^n).$$

From Proposition 6.7, by induction we get $\alpha^0 \leqslant \alpha^1 \leqslant \cdots$. We further let

$$\alpha^S = \bigsqcup_n \alpha^n.$$

For the sake of readability, the dependence on θ of α^S shall be tacitly understood. By Proposition 6.9 we immediately obtain

$$\alpha^S = T_S^\theta(\alpha^S). \tag{24}$$

Theorem 6.11. *Let $0 < \theta \leqslant 1$ with $\theta \in \mathbf{R}$. Let S be a set of nonnegative Horn clauses. Then $\alpha^S = \min(\text{Mod}_\theta S)$.*

Proof. By (24) together with Proposition 6.8, $\alpha^S \models_\theta S$. Let β be any θ-model of S, with the intent of proving $\alpha^S \leqslant \beta$. It is sufficient to show $\alpha^n \leqslant \beta$ for each $n = 0, 1, 2, \ldots$. We proceed by induction on n. The basis is trivial. For the induction step, assume $\alpha^n \leqslant \beta$. Then by monotonicity of T_S^θ together with Proposition 6.8 we have $\alpha^{n+1} = T_S^\theta(\alpha^n) \leqslant T_S^\theta(\beta) \leqslant \beta$, as required to complete the proof. □

Remark. With obvious modifications, the results obtained so far still hold in case uncountably many variable symbols are available, and $S = \{C_\alpha \mid \alpha < \kappa\}$ is an uncountable set of nonnegative Horn clauses. Far reaching generalizations of the model building techniques described so far in this section are discussed in [12] (see also [1]).

When θ is rational and S is finite, the least θ-model α^S can be quickly computed, as shown by the following result.

Theorem 6.12. *Let θ be a rational number $0<\theta=c/d\leqslant 1$, where c and d are integers in binary notation, and $1\leqslant c\leqslant d$. Let S be a finite set of nonnegative Horn clauses. Let $X_1,\ldots,X_n$ be the variables occurring in S. Let $|S|$ be the number of elements of S, and $\|S\|$ be the length of S. It follows that*

(i) *α^S coincides with α^t, for some integer $0\leqslant t\leqslant |S|$.*

(ii) *Further, there is a polynomial $\varphi=\varphi(x,y)$ such that, as an n-tupla $(a_1/b_1,\ldots, a_n/b_n)$ of pairs of relatively prime integers in binary notation, α^S is computable by a Turing machine over input (S,c,d) in less than $\varphi(\log d,\|S\|)$ steps.*

Proof. (i) For each $i=0,1,2,\ldots,$ let us define the set of clauses $\nabla^i\subseteq S$ by

$$\nabla^i=\{K\in S\mid \mathbf{K}_{\mathrm{Body}}(\alpha^i)>1-\theta \text{ and } \mathbf{K}_{\mathrm{Head}}(\alpha^i)<\theta\}.$$

Claim 1. For all $i\neq i'$, $\nabla^i\cap\nabla^{i'}=\emptyset$.

As a matter of fact, assume $K\in\nabla^i$. Let $P=P(X_j)$ be the head of K. Then $P\in\mathscr{A}(S, X_j,\theta,\alpha^i)$. From $\mathbf{P}(\alpha^i)<\theta$ we get $\mathrm{minsupp}_\theta P>\alpha^i_j$, whence, by definition of T^θ_S, for each integer $m>0$ we have $\alpha_j^{i+m}\geqslant\mathrm{minsupp}_\theta P$. It follows that $\mathbf{P}(\alpha^{i+m})\geqslant\theta$, and $K\notin\nabla^{i+m}$, as required to settle our claim.

One now similarly proves the following:

$$\nabla^i=\emptyset \quad \text{iff} \quad \alpha^i=\alpha^{i+1}, \tag{25}$$

whence by Claim 1 there must be an integer $0\leqslant t\leqslant |S|$ such that $\alpha^t=\alpha^{t+1}$. It follows from Theorem 6.11 that $\alpha^t=\alpha^S$ is the least θ-model of S, as required to prove our first statement.

(ii) Write S as in (23), and let $P_1,\ldots,P_m$ be the list of (positive) literals occurring in S. For each $u=1,\ldots,m$ let (a_u,b_u) be the pair of relatively prime integers $\geqslant 1$ such that $a_u/b_u=\mathrm{minsupp}_\theta P_u$. By Theorem 3.4,

$$a_u\leqslant b_u\leqslant d\|P_u\|\leqslant d\|S\|, \tag{26}$$

whence the length of the binary representation of the integers a_u and b_u is $\leqslant\log d$; as a matter of fact, the list of rational numbers $a_1/b_1,\ldots,a_m/b_m$ can be computed in a number of steps bounded by a polynomial in $\log d$ and $\|S\|$. Let us write in short, $\sigma_u=a_u/b_u$. Arguing as in the proof of the claim in Theorem 3.4, we see that the list

$$(K,\mathbf{K}_{\mathrm{Body}}(\sigma_u),\mathbf{K}_{\mathrm{Head}}(\sigma_u))$$

(for all $K \in S$ and $u = 1, \ldots, m$) can be computed in a number of steps bounded by a polynomial in $\log d$ and $\|S\|$. By Definition 6.6, for every $i = 0, \ldots, t$ and $j = 1, \ldots, n$, the jth coordinate of α^i is either zero, or else belongs to the set $\{\sigma_0, \ldots, \sigma_m\}$. There is a Turing machine which, over any input (α^i, S, c, d), outputs α^{i+1} in a number of steps bounded by a polynomial in $\log d$ and $\|S\|$. Since $t \leqslant |S| \leqslant \|S\|$, by (26) we conclude that the list $\alpha^0, \ldots, \alpha^t$ is computable by a Turing machine over input (S, c, d), in a number of steps bounded by a polynomial in $\log d$ and $\|S\|$. This completes the proof. $\square$

Corollary 6.13. *There is a polynomial β in two variables, and a Turing machine $\mathscr{T}$ such that for every rational $0 < \theta = c/d \leqslant 1$ (where $1 \leqslant c \leqslant d$ are integers in binary notation) and every finite set S of Horn clauses, $\mathscr{T}$ over input (S, c, d) decides whether S is θ-satisfiable, in a number of steps smaller than $\beta(\log d, \|S\|)$. Thus, the $HORNSAT_\infty$ problem is decidable in polynomial time.*

Proof. Assume $X_1, \ldots, X_m$ is the list of variables occurring in S.

Method 1. Let S^+ be the set of nonnegative clauses in S. By Theorem 6.12 one can quickly compute the least θ-model $\alpha = (\alpha_1, \ldots, \alpha_m)$ of S^+, where each coordinate α_i is a pair of relatively prime integers in binary notation. Arguing as in the proof of the claim in Theorem 3.4, we can quickly check whether for every $C \in S$, $\mathbf{C}(\alpha) \geqslant \theta$.

Method 2. Let $Y_\perp$ be a variable symbol not occurring in S. Let $S^\perp$ be obtained by replacing each negative clause $C \in S$ by the nonnegative clause $C^\perp = C \cup \{Y_\perp\}$. Thus, whenever $C = \{\neg P_1, \ldots, \neg P_k\}$, it follows that $C^\perp = P_1 \wedge \cdots \wedge P_k \Rightarrow Y_\perp$. Then it is not hard to see that S is θ-unsatisfiable iff $Y_\perp$ takes the value θ in the least model $\alpha^{S^\perp}$ of $S^\perp$. By Theorem 6.12 the computation of $\alpha^{S^\perp}$, as a list of pairs of relatively prime integers in binary notation, can be computed in a number of Turing steps bounded by a polynomial in d and $\|S\|$. $\square$

Remark. As an application of the above corollary, let S be a finite and satisfiable set of Horn clauses; let $\theta \in \mathbf{Q}$ be the maximum value such that there exists $\beta \models_\theta S$. Then for any integer k such that $2^{-k} < \theta$ one can quickly (i.e., polynomially in k and $\|S\|$) find an assignment $\alpha \models_{\theta - 2^{-k}} S$.

7. The expressive power of literals

In this section we shall prove the converse of Corollary 2.4. By analogy with Eqs. (4)–(5), the binary operations $\oplus$ and $\odot$ on the real unit interval $[0,1]$ are defined by

$$x \oplus y = \min(1, x + y) \quad \text{and} \quad x \odot y = \max(0, x + y - 1).$$

Further, the function ${}^\# : \mathbf{R} \to [0,1]$ is defined by stipulating that for all $z \in \mathbf{R}$

$$z^\# = \max(0, \min(1, z)).$$

Proposition 7.1. *For all $z \in \mathbf{R}$ and $x \in [0,1]$ we have the identity*

$$(z+x)^{\#} = (z^{\#} \oplus x) \odot (z+1)^{\#}. \tag{27}$$

Proof. If $z \leqslant -1$ both sides are equal to 0. If $z > 1$ both sides are equal to 1. If $z \in [0,1]$ both sides are equal to $z \oplus x$. Finally, if $-1 < z < 0$ we have $(z^{\#} \oplus x) \odot (z+1)^{\#} = x \odot (z+1) = \max(0, x+z) = (z+x)^{\#}$. □

Proposition 7.2. *Let $0 \leqslant b < a$ be integers. Let the linear function $g : [0,1] \to \mathbf{R}$ be defined by $g(x) = ax - b$ for all $x \in [0,1]$. Then there is a positive literal $P = P(X)$ only containing the connectives $\odot$ and $\oplus$, such that for all $x \in [0,1]$,*

$$g(x)^{\#} = \mathbf{P}(x). \tag{28}$$

Proof. By induction on a. If $a = 1$ then let $P = X$. Otherwise, recalling (27), for all $x \in [0,1]$ we have the identity

$$((a-1)x - b + x)^{\#} = (((a-1)x - b)^{\#} \oplus x) \odot ((a-1)x - (b-1))^{\#}. \tag{29}$$

If $a - 1 > b > 0$ then the desired conclusion immediately follows from (29) by induction hypothesis. If $a - 1 = b > 0$ then the right-hand side of (29) becomes equal to $x \odot ((a-1)x - (b-1))^{\#}$, and again, the result follows by induction. Finally, if $a - 1 > b = 0$, the right-hand side of (29) is equal to $((a-1)x)^{\#} \oplus x$, whence, also in this case, the desired result follows by induction. □

By Definition 2.2 there are precisely two constant McNaughton functions $f : [0,1] \to [0,1]$, namely $f = 0$ and $f = 1$.

Lemma 7.3. *Let $f : [0,1] \to [0,1]$ be a monotone increasing McNaughton function other than the constants 0 and 1. Then*

(i) *There are uniquely determined rationals $0 \leqslant \alpha < \beta \leqslant 1$ such that $f = 0$ over $[0, \alpha]$, $f = 1$ over $[\beta, 1]$, and f is strictly increasing over $[\alpha, \beta]$.*

(ii) *There is a positive literal $P = P^0$ and a rational $\gamma > \alpha$ such that $\mathbf{P} = f$ over $[0, \gamma]$ and $\mathbf{P} \geqslant f$ over $[\gamma, 1]$.*

(iii) *There is a positive literal $P = P^1$ and a rational $\delta < \beta$ such that $\mathbf{P} = f$ over $[\delta, 1]$ and $\mathbf{P} \geqslant f$ over $[0, \delta]$.*

(iv) *For all $\xi \in \mathbf{R}$ with $\alpha < \xi < \beta$ there is an open neighbourhood $\mathcal{N}_\xi$ of ξ and a positive literal $P = P^\xi$ such that $\mathbf{P} = f$ over $\mathcal{N}_\xi$, and $\mathbf{P} \geqslant f$ over the complementary set $[0,1] \backslash \mathcal{N}_\xi$.*

Proof. (i) With reference to Definition 2.2, let $p_0, p_1, \ldots, p_w, p_{w+1}$ be the polynomials with integer coefficients representing the linear pieces of f, say without loss of generality

$$p_0 = 0, \quad p_{w+1} = 1, \quad p_j = a_j x - b_j, \qquad 0 \leqslant b_j < a_j, \tag{30}$$

for each $j=1,\dots,w$. Then by our hypotheses about f, we immediately obtain the desired conclusion.

(ii) There is an index $j_0\in\{1,\dots,w\}$ and a rational $\gamma>\alpha$ such that f coincides with the polynomial p_{j_0} on $[\alpha,\gamma]$. Further, there is a linear polynomial $q(x)=ax-b$ with integer coefficients $0\leqslant b<a$ such that $q(\gamma)=f(\gamma)$ and $a>\max(a_1,\dots,a_w)$. It follows that $q(x)<f(x)$ for all $0\leqslant x<\gamma$, and $q(x)>f(x)$ for all $\gamma<x\leqslant 1$. By Proposition 7.2 there are positive literals L and M in the same variable X, only containing the connectives $\oplus$ and $\odot$, such that $\mathbf{L}=p_{j_0}^{\#}$ and $\mathbf{M}=q^{\#}$. Then the positive literal $P=L\vee M$ satisfies the identity $\mathbf{P}=(p_{j_0}\vee q)^{\#}$ on $[0,1]$, and also satisfies the requirements of (ii).

(iii) There is an index $i_0\in\{1,\dots,w\}$ and a rational $\delta<\beta$ such that $f=p_{i_0}$ on the left neighbourhood $\mathcal{N}_0^{\leftarrow}=[\delta,\beta]$ of β. If $f\leqslant p_{i_0}$ on the whole $[0,1]$ we are done. Otherwise, let $0<\delta_1\leqslant\delta$ be the rightmost point such that $p_{i_0}(\delta_1)=f(\delta_1)$ and $p_{i_0}(x)<f(x)$ for all $x<\delta_1$ sufficiently close to δ_1. It follows that $\delta_1\in\mathbf{Q}$, and there is an index $i_1\in\{0,\dots,w\}$ such that $f=p_1$ on a left neighbourhood $\mathcal{N}_1^{\leftarrow}$ of δ_1. In the light of (30) we must have $a_{i_0}>a_{i_1}\geqslant 0$. If $f\leqslant p_{i_1}$ over $[0,\delta_1]$ we are done. Otherwise $a_{i_1}>0$ and, proceeding by induction, we have a (necessarily finite) sequence of indexes $i_0,i_1,\dots,i_t\in\{0,\dots,w\}$, together with integers $a_{i_0}>a_{i_1}>\cdots>a_{i_t}\geqslant 0$ and rational points $\beta=\delta_0>\delta_1>\delta_2>\cdots>\delta_t>\delta_{t+1}=0$ such that for each $j=0,\dots,t$, $f\leqslant p_{i_j}$ on $[\delta_{j+1},\delta_j]$, and $f=p_{i_j}$ on a left neighbourhood $\mathcal{N}_j^{\leftarrow}$ of δ_j. It follows that $f=p_{i_0}^{\#}\vee\cdots\vee p_{i_t}^{\#}$ on $[\delta,\beta]$, and $f\leqslant p_{i_0}^{\#}\vee\cdots\vee p_{i_t}^{\#}$ over $[0,1]\backslash[\delta,\beta]$. Again by Proposition 7.2 there are literals $L_0,\dots,L_t$, all in the same variable X, and only containing the connectives $\oplus$ and $\odot$ such that $\mathbf{L}_r=p_{i_r}^{\#}$ for each $r=0,\dots,t$. Therefore, the literal $P=L_0\vee\cdots\vee L_t$ satisfies the requirements of (iii).

(iv) We shall consider two cases:

Case 1: *f is differentiable at ξ*. Then for some index $j=1,\dots,w$ the function f coincides with p_j on a closed neighbourhood $\mathcal{N}=[\rho,\sigma]$ of ξ, for suitable rationals $\alpha<\rho<\xi<\sigma<\beta$. Arguing as in (ii) there is a linear polynomial $q(x)=ax-b$ (for some integers $0\leqslant b<a$) such that $q(\sigma)=f(\sigma)$, $a>\max(a_1,\dots,a_w), q(x)>f(x)$ for all $x>\sigma$ and $q(x)<f(x)$ for all $x<\sigma$. Arguing as in the second part of (iii) there are indexes $i_1,\dots,i_t\in\{0,\dots,w\}$ such that $f\leqslant p_{i_1}^{\#}\vee\cdots\vee p_{i_t}^{\#}$ on $[0,\rho]$. Let the function $h:[0,1]\to[0,1]$ be defined by $h=p_{i_1}^{\#}\vee\cdots\vee p_{i_t}^{\#}\vee p_j^{\#}\vee q^{\#}$. Then $f\leqslant h$ on $[0,1]$ and $f=h$ on $[\rho,\sigma]$. An application of Proposition 7.2 yields literals $L_1,\dots,L_t,A,B$, all in the same variable X, and only containing the connectives $\odot$ and $\oplus$, such that $\mathbf{L}_1=p_{i_1}^{\#},\dots,\mathbf{L}_t=p_{i_t}^{\#}$, $\mathbf{A}=p_j^{\#}$, $\mathbf{B}=q^{\#}$. Therefore, the literal $P=L_1\vee\cdots\vee L_t\vee A\vee B$ has the desired properties.

Case 2: *f is not differentiable at ξ*. Then let a' and a'' be the left and the right derivative of f at ξ. By (30), $a',a''\in\{a_1,\dots,a_w\}$, and there are indexes $j',j''\in\{1,\dots,w\}$ and rationals $\rho<\xi<\sigma$ such that, over the interval $[\rho,\sigma]$, the function f either coincides with $p_{j'}\vee p_{j''}$ (in case $a'<a''$), or with $p_{j'}\wedge p_{j''}$ (in case $a'>a''$). Arguing as in Case 1, there are polynomials $q,p_{i_1},\dots,p_{i_t}$ such that, letting the functions h and k be defined by $h=p_{i_1}^{\#}\vee\cdots\vee p_{i_t}^{\#}\vee p_{j'}^{\#}\vee p_{j''}^{\#}\vee q^{\#}$ and $k=p_{i_1}^{\#}\vee\cdots\vee p_{i_t}^{\#}\vee(p_{j'}^{\#}\wedge p_{j''}^{\#})\vee q^{\#}$, it follows that either f coincides with h or with k on $[\rho,\sigma]$, and $f\leqslant h$, $f\leqslant k$ on $[0,1]\backslash[\rho,\sigma]$. By Proposition 7.2 there are literals $L_1,\dots,L_t,L',L'',T$ all in the same variable X, and only containing the connectives $\odot$ and $\oplus$, such that $\mathbf{L}_1 = p^{\#}_{i_1}, \ldots, \mathbf{L}_t = p^{\#}_{i_t}$, $\mathbf{L}' = p^{\#}_{j'}, \mathbf{L}'' = p^{\#}_{j''}, \mathbf{T} = q^{\#}$. In conclusion, either literal $L_1 \vee \cdots \vee L_t \vee L' \vee L'' \vee T$ or $L_1 \vee \cdots \vee L_t \vee (L' \wedge L'') \vee T$ has the desired properties. □

Theorem 7.4. *If $f:[0,1] \to [0,1]$ is a monotone increasing (resp., decreasing) McNaughton function, other than the constants 0 and 1, then there is a positive (resp., negative) literal $L = L(X)$ such that $\mathbf{L} = f$.*

Proof. We shall only consider the case when f is increasing. Let α and β be as in Lemma 7.3. Then for each $\xi \in [\alpha, \beta]$ there is an open neighbourhood $\mathcal{N}_\xi$ of ξ in $[0,1]$ and a positive literal $P^\xi = P^\xi(X)$ such that $f = \mathbf{P}^\xi$ on $\mathcal{N}_\xi$ and $f \leqslant \mathbf{P}^\xi$ on $[0,1]$. By compactness there are finitely many points $\xi(1), \ldots, \xi(u)$ such that $\mathcal{N}_{\xi_1} \cup \cdots \cup \mathcal{N}_{\xi_u} \supseteq [\alpha, \beta]$. Without loss of generality we can assume $\xi(1) = \alpha$ and $\xi(u) = \beta$, whence $\mathbf{P}^{\xi(1)}(x) = 0$ for all $0 \leqslant x \leqslant \alpha$. By the same lemma, the literal $L = P^{\xi(1)} \wedge \cdots \wedge P^{\xi(u)}$ satisfies the requirements of the theorem. □

Remark. Both Propositions 7.1 and 7.2 are due to McNaughton [5]. Our present proof of these two results is a simplification of the proof of Rose and Rosser [10]. It is an open problem whether every nonconstant monotone increasing McNaughton function f of one variable is representable by a literal $L = L(X)$ only containing the connectives $\odot$ and $\oplus$.[3]

As anticipated in an earlier section, the following result shows that the connectives $\oplus$ and $\odot$ *alone* cannot express the lattice connectives $\wedge$ and $\vee$ (compare with (8)).

Proposition 7.5. *There is no formula $F = F(X, Y)$ only containing the connectives $\oplus$ and $\odot$ such that $\mathbf{F}(x, y) = \max(x, y)$.*

Proof.[4] Otherwise (absurdum hypothesis) let F be such formula. Replacing Y by X in $F(X, Y)$ we get a formula $D(X) \neq X$ such that $\mathbf{D}(x) = x$ for all $x \in [0,1]$. Looking at the outermost connective in D there are two cases to consider:

Case 1: $D = (G \odot H)$ *for suitable positive literals $G(X)$ and $H(X)$.* Since both $\mathbf{G}$ and $\mathbf{H}$ are continuous functions vanishing at $x = 0$, it follows that $\mathbf{D}(x) = 0$ for all suitably small $x \geqslant 0$, a contradiction.

Case 2: $D = (G \oplus H)$ *for suitable positive literals $G(X)$ and $H(X)$.* Then let a' and a'' be the right derivatives of $\mathbf{G}$ and $\mathbf{H}$ at 0. By Proposition 2.3 together with our hypothesis, the integers $0 \leqslant a'$ and $0 \leqslant a''$ satisfy the condition $a' + a'' = 1$. We can safely assume $a' = 0$ and $a'' = 1$. Since $\mathbf{G}(1) = 1$, by elementary calculus there must be a point $0 < \xi < 1$ such that $(\mathrm{d}/\mathrm{d}x)\mathbf{G}(\xi) \geqslant 2$. We can safely assume that also $\mathbf{H}$ is

[3] A positive answer is given by S. Aguzzoli in his paper "The complexity of McNaughton functions of one variable", to appear in *Advances in Applied Mathematics*.

[4] The referee kindly supplied an alternative and shorter proof, showing, by induction on $\|F\|$, that there is no formula $F = F(X, Y)$ only containing the connectives $\oplus$ and $\odot$ such that $\mathbf{F}(0, 1/2) = \mathbf{F}(1/2, 0) = \mathbf{F}(1/2, 1/2) = 1/2$.

differentiable at ξ. Since $\mathbf{D}(x)=x$ we get the contradiction

$$1=\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{D}(\xi)=\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{G}\oplus\mathbf{H})(\xi)=\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{G}+\mathbf{H})(\xi)\geqslant 2. \qquad \square$$

8. Further research

The fact that every formula in the classical propositional calculus is equivalent to a finite set of clauses is no longer true in the infinite-valued calculus of Łukasiewicz. Nevertheless, in [8] it is shown that *any* formula F does have a natural disjunctive normal form, whence also a conjunctive normal form reduction in the infinite-valued calculus. While Farey mediants still maintain a fundamental role, it turns out that the manipulations on the general normal forms of [8] are more akin to toric desingularization than to our present resolution procedures.

In want of a general theory of automated deduction in the infinite-valued calculus, our present literals, and their associated resolution procedures, strike a balance between expressive power and computational costs. One can reasonably expect that the polynomial time algorithms introduced in this paper may find some applications in the logical treatment of propositions taking infinitely many truth-values.

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